Abstract. We prove that many important weak double categories can be ‘repre-
sented’ by spans, using the basic higher limit of the theory: the tabulator. Dually,
representations by cospans via cotabulators are also frequent.

Introduction

Strict double categories were introduced and studied by C. Ehresmann [Eh1, Eh2], the
weak notion in our series [GP1 - GP4]. The strict case extends the more usual (if histor-
ically subsequent) notion of 2-category, while the weak one extends bicategories, priorly
established by Bénabou [Be]. The extension is made clear in Section 3.

This note is about weak double categories and the (horizontal) tabulator
of a vertical
arrow. The latter is the ‘basic’ higher limit of the theory; in fact the main result of [GP1]
says that a weak double category has all (horizontal) double limits if and only if it has:
double products, double equalisers and tabulators.

We prove here that the existence of tabulators in a weak double category \( A \) produces,
under suitable hypotheses, a lax functor \( S: A \to \text{Span}(C) \) with values in the weak double
category of spans over the category \( C \) of horizontal arrows of \( A \) (Theorem 6). We say
that \( A \) is span representable when this functor \( S \) is horizontally faithful.

Many important weak double categories can be represented in this sense, by spans or
- dually - by cospans, via cotabulators.

Outline. We begin by a brief review of basic notions on weak double categories, from [GP1,
GP2], including the weak double categories of spans and cospans, and the (co)tabulator
of a vertical arrow.

Sections 6 and 7 give the main definitions and results recalled above, about (co)span
representability. Various weak (or strict) double categories are examined in Sections 8 -
12, proving that many of them are both span and cospan representable. Yet the weak
double category \( \text{SpanSet} \), which is trivially span representable, is not cospan representable
(Section 8), and \( \text{CospSet} \) behaves in a dual way.

Finally, some common patterns in the previous proofs of representability are analysed
in Section 13.

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1. Definition

A (strict) double category $\mathcal{A}$ consists of the following structure.

(a) A set $\text{Ob}\mathcal{A}$ of objects of $\mathcal{A}$.

(b) Horizontal morphisms $f: X \to X'$ between the previous objects; they form the category $\text{Hor}_0\mathcal{A}$ of the objects and horizontal maps of $\mathcal{A}$, with composition written as $gf$ and identities $1_X: X \to X$.

(c) Vertical morphisms $u: X \to Y$ (often denoted by a dot-marked arrow) between the same objects; they form the category $\text{Ver}_0\mathcal{A}$ of the objects and vertical maps of $\mathcal{A}$, with composition written as $vu$ (or $u \otimes v$, in diagrammatic order) and identities written as $e_X: X \to X$ or $1^*_X$.

(d) Double cells $a: (u \xymatrix{f & g})$ with a boundary formed of two vertical arrows $u, v$ and two horizontal arrows $f, g$.

(e) Cells have a horizontal composition, consistent with the horizontal composition of arrows and written as $(a | b): (u \xymatrix{f & g}) w$, or $a|b$; this composition gives the category $\text{Hor}_1\mathcal{A}$ of vertical arrows and cells $a: u \to v$ of $\mathcal{A}$, with identities $1_u: (u 1^* u)$.

(f) Cells have also a vertical composition, consistent with the vertical composition of arrows and written as $(\xi \xymatrix{a & c}) : (u' \xymatrix{v' & v}) w'$, or $\xi | a$; this composition gives the category $\text{Ver}_1\mathcal{A}$ of horizontal arrows and cells $a: f \to g$ of $\mathcal{A}$, with identities $e_f = 1^*_g: (e_f e)$.
The two compositions satisfy the interchange laws (for binary and zeroary compositions), which means that we have, in diagram (2):
\[
\begin{align*}
\begin{array}{c}
\begin{array}{c} a \mid b \\
\end{array}
\end{array}
\end{align*}
\begin{array}{c}
\begin{array}{c} c \mid d \\
\end{array}
\end{array}
\right) =
\left(\begin{array}{c}
\begin{array}{c} a \mid b' \\
\end{array}
\end{array}
\right)
\left(\begin{array}{c}
\begin{array}{c} 1_u \mid 1_{u'} \\
\end{array}
\end{array}
\right) = 1_{u' \ast u}, \\
(e_f \mid e_{f'}) = e_{f'f}, \\
1_{e_X} = e_{1_X}.
\end{align*}
(3)

The first condition says that a consistent matrix \((a \mid b)\) has a precise pasting; the last says that an object \(X\) has an identity cell \(\square_X = 1_{e_X} = e_{1_X}\). The expressions \((a \mid f')\) and \((f \mid b)\) will stand for \((a \mid e_{f'})\) and \((e_f \mid b)\), when this makes sense.

\(A\) is said to be flat if every double cell \(a: (u \mid f \mid v)\) is determined by its boundary - namely the arrows \(f, g, u, v\). A standard example is the double category \(\mathbb{RelSet}\) of sets, mappings and relations, recalled below in Section 8(c).

2. Hints at weak double categories

More generally, in a weak double category \(A\) the horizontal composition behaves categorically (and we still have ordinary categories \(\text{Hor}_0A\) and \(\text{Hor}_1A\)), while the composition of vertical arrows is categorical up to comparison cells:
- for a vertical arrow \(u: X \to Y\) we have a left unitor and a right unitor
  \[
  \lambda u: e_X \otimes u \to u, \quad \rho u: u \otimes e_Y \to u,
  \]
- for three consecutive vertical arrows \(u: X \to Y, v: Y \to Z\) and \(w: Z \to T\) we have an associator
  \[
  \kappa(u, v, w): u \otimes (v \otimes w) \to (u \otimes v) \otimes w.
  \]

Interchange holds strictly, as above. The comparison cells are special (which means that their horizontal arrows are identities) and horizontally invertible. Moreover they are assumed to be natural and coherent, in a sense made precise in [GP1], Section 7; after stating naturality with respect to double cells, the coherence axioms are similar to those of bicategories.

\(A\) is said to be unitary if the unitors are identities, so that the vertical identities behave as strict units - a constraint which in concrete cases can often be easily met. The terminology of the strict case is extended to the present one, as far as possible.

A lax (double) functor \(F: X \to A\) between weak double categories amounts to assigning:

(a) two functors \(\text{Hor}_0F\) and \(\text{Hor}_1F\), consistent with domain and codomain

\[
\begin{array}{ccc}
\text{Hor}_1X & \xrightarrow{\text{Hor}_1F} & \text{Hor}_1A \\
\text{Dom} \downarrow & & \downarrow \text{Dom} \\
\text{Hor}_0X & \xrightarrow{\text{Hor}_0F} & \text{Hor}_0A
\end{array}
\]

\[
\begin{array}{ccc}
\text{Hor}_1X & \xrightarrow{\text{Hor}_1F} & \text{Hor}_1A \\
\text{Cod} \downarrow & & \downarrow \text{Cod} \\
\text{Hor}_0X & \xrightarrow{\text{Hor}_0F} & \text{Hor}_0A
\end{array}
\] (4)
(b) for any object \( X \) in \( \mathbb{X} \), a special cell, the \textit{identity comparison} of \( F \)
\[
F(X): e_F X \to F e_X : FX \to FX,
\]
(c) for any vertical composite \( u \otimes v: X \to Y \to Z \) in \( \mathbb{X} \), a special cell, the \textit{composition comparison} of \( F \)
\[
F(u, v): Fu \otimes Fv \to F(u \otimes v): FX \to FZ.
\]
Again, these comparisons must satisfy axioms of naturality and coherence with the comparisons of \( \mathbb{X} \) and \( \mathbb{A} \) [GP2].

3. Dualities

A weak double category has a \textit{horizontal opposite} \( \mathbb{A}^h \) (reversing the horizontal direction) and a \textit{vertical opposite} \( \mathbb{A}^v \) (reversing the vertical direction); a strict structure also has a \textit{transpose} \( \mathbb{A}^t \) (interchanging the horizontal and vertical issues).

The prefix ‘co’, as in \textit{colimit}, \textit{coequaliser} or \textit{colax double functor}, refers to horizontal duality, the main one. Let us note that a weak double category whose horizontal arrows are identities is the same as a \textit{bicategory written in vertical}, i.e. with arrows and weak composition in the vertical direction and strict composition in the horizontal one. This is why the oplax functors of bicategories correspond here to colax double functors. (Transposing the theory of double categories, as is done in some papers, would avoid this conflict of terminology, but would produce other conflicts at a more basic level: for instance, colimits in \( \textbf{Set} \) would become ‘op-limits’ in \( \textbf{RelSet} \) and \( \textbf{SpanSet} \).)

4. Spans and cospans

For a category \( \mathbf{C} \) with (a fixed choice of) pullbacks there is a weak double category \( \text{Span}(\mathbf{C}) \) of spans over \( \mathbf{C} \), which will play here an important role.

Objects, horizontal arrows and their composition come from \( \mathbf{C} \), so that \( \text{Hor}_0(\text{Span}\mathbf{C}) = \mathbf{C} \).

A vertical arrow \( u: X \to Y \) is a span \( u = (u', u'') \), i.e. a diagram \( X \leftarrow U \to Y \) in \( \mathbf{C} \), or equivalently a functor \( u: \mathcal{V} \to \mathbf{C} \) defined on the formal-span category \( \mathcal{V} \leftarrow \mathcal{V} \rightarrow \mathcal{V} \).

A vertical identity is a pair \( e_X = (1_X, 1_X) \). A cell \( \sigma: (u \overset{f}{\to} v) \) is a natural transformation \( u \to v \) of such functors and amounts to the left commutative diagram below

\[
\begin{array}{ccc}
X & \overset{f}{\to} & X' \\
\downarrow u' & & \downarrow v' \\
U & \xrightarrow{m\sigma} & V \\
\downarrow u'' & & \downarrow v'' \\
Y & \overset{g}{\to} & Y'
\end{array}
\quad
\begin{array}{ccc}
X & \overset{f}{\to} & X' & \overset{f'}{\to} & X'' \\
\downarrow u' & & \downarrow v' & & \downarrow w' \\
U & \xrightarrow{m\sigma} & V & \xrightarrow{m\tau} & W \\
\downarrow u'' & & \downarrow v'' & & \downarrow w'' \\
Y & \overset{g}{\to} & Y' & \overset{g'}{\to} & Y''
\end{array}
\]
We say that the cell \( \sigma \) is represented by its middle arrow \( m\sigma : U \to V \), which determines it together with the boundary (the present structure is not flat).

The horizontal composition \( \sigma \mid \tau \) of \( \sigma \) with a second cell \( \tau : v \to w \) is a composition of natural transformations, as in the right diagram above; it gives the category \( \text{Hor}_1(\text{Span}C) = \text{Cat}(\land, C) \).

The vertical composition \( u \otimes v \) of spans is computed by (chosen) pullbacks in \( C \)

\[
\begin{array}{ccc}
X & \xleftarrow{U} & Y \\
\downarrow & & \downarrow \\
W & \xrightarrow{V} & Z
\end{array}
\]

\[ W = U \times_Y V. \tag{6} \]

This is extended to double cells, in the obvious way. For the sake of simplicity we make \( \text{Span}(C) \) unitary, by adopting the ‘unit constraint’ for pullbacks: the chosen pullback of an identity along any morphism is an identity. The associator \( \kappa \) is determined by the universal property of pullbacks.

Dually, for a category \( C \) with (a fixed choice of) pushouts there is a unitary weak double category \( \text{Cosp}(C) \) of cospans over \( C \), that is horizontally dual to \( \text{Span}(C^{op}) \). We have now

\[
\text{Hor}_0(\text{Cosp}C) = C, \quad \text{Hor}_1(\text{Cosp}C) = \text{Cat}(\land, C), \tag{7}
\]

where \( \land = \land^{op} \) is the formal-cospan category \( \bullet \to \bullet \leftarrow \bullet \).

A vertical arrow \( u = (u', u'') : \land \to C \) is now a cospan, i.e. a diagram \( X \to U \leftarrow Y \) in \( C \), and a cell \( \sigma : u \to v \) is a natural transformation of such functors. Their vertical composition is computed with pushouts in \( C \); again, we generally follow the ‘unit constraint’ for pushouts.

5. Tabulators

The (horizontal) tabulator of a vertical arrow \( u : X \to Y \) in the weak double category \( A \) is an object \( T = \top u \) equipped with a double cell \( t_u : e_T \to u \)

\[
\begin{array}{ccc}
T & \xrightarrow{p} & X \\
\downarrow e_T & & \downarrow u \\
T & \xrightarrow{q} & Y
\end{array}
\quad
\begin{array}{ccc}
H & \xrightarrow{f} & T \\
\downarrow e_f & & \downarrow t_u \\
H & \xrightarrow{f} & T
\end{array}
\quad
\begin{array}{ccc}
H & \xrightarrow{f} & T \\
\downarrow e_f & & \downarrow t_u \\
H & \xrightarrow{f} & T \\
\downarrow e_f & & \downarrow t_u \\
H & \xrightarrow{f} & T \\
\downarrow e_f & & \downarrow t_u \\
H & \xrightarrow{f} & T
\end{array}
\]

such that the pair \( (T, t_u : e_T \to u) \) is a universal arrow from the functor \( e : \text{Hor}_0A \to \text{Hor}_1A \) to the object \( u \) of \( \text{Hor}_1A \). Explicitly, this means that for every object \( H \) and every cell \( h : e_H \to u \) there is a unique horizontal map \( f : H \to T \) such that \( (e_f \mid t_u) = h \), as in the right diagram above.

(In [GP2] we also considered a higher dimensional universal property, which was dropped in later papers and is not used here.) We say that \( A \) has tabulators if all of
them exist, or equivalently if the degeneracy functor \( e : \text{Hor}_0 A \to \text{Hor}_1 A \) has a right adjoint

\[
\top : \text{Hor}_1 A \to \text{Hor}_0 A, \quad e \dashv \top.
\]  

(9)

In this situation one can try to represent \( A \) as a weak double category of spans, as we shall see below.

Dually \( A \) has cotabulators if the degeneracy functor has a left adjoint \( \perp : \text{Hor}_1 A \to \text{Hor}_0 A, \quad \perp \dashv e \),

(10)

so that every vertical arrow \( u : X \to Y \) has a cotabulator-object \( \perp u \), equipped with two horizontal morphisms \( i : X \to \perp u \), \( j : Y \to \perp u \) and a universal cell \( \iota : (u^i_j e) \). This may allow representing \( A \) as a weak double category of cospans.

For a category \( C \) with pullbacks, the tabulator in \( \text{Span}(C) \) of a span \( u = (u', u'') = (X \leftarrow U \to Y) \) is its central object \( U \), with projections \( u', u'' \) and the obvious cell \( t_u : e_U \to u \). The cotabulator is the pushout of the span in \( C \), provided it exists. All this cannot be formulated within the bicategory \( \text{Span}(C) \) (vertically embedded in \( \text{Span}(C) \) as specified in Section 3).

6. Theorem and Definition (Span representation)

We suppose that:

(a) the weak double category \( A \) has tabulators,

(b) the ordinary category \( C = \text{Hor}_0(A) \) of objects and horizontal arrows has pullbacks.

There is then a canonical lax functor, which is trivial in degree zero

\[
S : A \to \text{Span}(C), \quad \text{Hor}_0(S) = \text{id}_C, \tag{11}
\]

and takes a vertical arrow \( u : X \to Y \) of \( A \) to the span \( Su = (p, q) : X \to Y \) determined by the tabulator \( \top u \) and its projections \( p : \top u \to X, \quad q : \top u \to Y \).

The lax functor \( S \) will be called the span representation of \( A \).

Note. Related results can be found in Niefield [Ni], for weak double categories with vertical companions and adjoints.

PROOF. As in Section 5 we write \( t_u : (e^p_q u) \) the universal cell of the tabulator \( \top u \).

The action of \( S \) on a cell \( a \) of \( A \) is described by the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow{u} & \searrow{a} & \nearrow{v} \\
Y & \xrightarrow{g} & Y'
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow{t_u} & \searrow{\top u} & \nearrow{t_v} \\
Y & \xrightarrow{g} & Y'
\end{array}
\]

(12)
where the cell $Sa: Su \to Sv$ (a morphism of spans) is represented by the coherent morphism $Ta: Tu \to Tv$. The latter is determined by the universal property of the universal cell $t_v$ of the tabulator $Tv$

$$
(Ta \mid t_v) = (t_u \mid a) \quad (p'.Ta = fp, \ q'.Ta = gq), \quad (13)
$$

(In the composition $(Ta \mid t_v)$ we write $Ta$ for $e_{Ta}$, as already warned at the end of Section 1.)

To define the laxity comparisons, an object $X$ of $A$ gives a special cell $S(X): e_X \to S(e_X)$ represented by the morphism

$$
k_X: X \to \top e_X, \quad (k_X \mid t_{eX}) = \square_X. \quad (14)
$$

For a vertical composite $w = u \otimes v: X \to Y \to Z$, the comparison $S(u, v): Su \otimes Sv \to Sw$ is represented by the morphism $k_{uv}$ defined below, where $P = \top u \times Y \top v$ is a pullback and $\sigma = \lambda(e_p)^{-1} = \rho(e_p)^{-1}$

$$
k_{uv}: P \to \top w, \quad (k_{uv} \mid t_w) = \left(\sigma \mid r \mid t_u \mid s \mid t_v\right), \quad (15)
$$

(Note that one cannot apply interchange to $(r \mid t_u) \otimes (s \mid t_v)$.) Finally we have to verify the coherence conditions of the comparisons of $S$ (see [GP2], Section 2.1), and we only check axiom (iii) for the right unitor.

For a vertical map $u: X \to Y$ and $w = u \otimes e_Y$ we have to verify that the following diagram of morphisms of $C$ commutes

$$
(16)
$$
where the pullback $\top u \times Y$ is realised as $\top u$, by the unit constraint, and the morphism $\top(\rho u)$ is defined by: $(\top(\rho u) | t_u) = (t_w | \rho u)$.

Equivalently, by applying the (cancellable) universal cell $t_u$ and the isocell $\rho = \rho(e_{\top u})$, we show that

$$(\rho | (1, k_Y) | k_{ue} | \top(\rho u) | t_u) = (\rho | t_u).$$

In fact we have

$$(\rho | (1, k_Y) | k_{ue} | \top(\rho u) | t_u) = (\rho | (1, k_Y) | k_{ue} | t_w | \rho u)$$

$$= \left(\frac{\rho | (1, k_Y) | \sigma | r | t_u | \rho u}{s | t_e} = \left(\frac{\rho | \rho^{-1} | (1, k_Y) | r | t_u | \rho u}{s | t_e} \right) \right)\ (17)$$

The fourth, fifth and seventh terms of these computations are represented below, with $P = \top u \times Y \top e_Y$ and $k = k_Y$

7. Span and cospan representability

Let $\mathcal{A}$ be a weak double category.

(a) We say that $\mathcal{A}$ is (horizontally) span representable if:
- it has tabulators,
- the ordinary category $C = \text{Hor}_0(A)$ has pullbacks,
- the span-representation lax functor $S: A \to \text{Span}(C)$ of (11) is horizontally faithful.

The last condition means that the ordinary functors $\text{Hor}_0 S$ and $\text{Hor}_1 S$ are faithful. This is trivially true for $\text{Hor}_0(S) = \text{id} C$, and also for $\text{Hor}_1 S$ when $A$ is flat.

(b) By horizontal duality, if $A$ has cotabulators and $C = \text{Hor}_0(A)$ has pushouts we form a colax functor of cospan representation

$$C: A \to \text{Cospan}(C), \quad \text{Hor}_0(C) = \text{id} C,$$

that takes a vertical arrow $u: X \to Y$ of $A$ to the cospan $Cu = (i,j): X \to Y$ formed by the cotabulator $\perp u$ and its ‘injections’ $i: A \to \perp u$, $j: Y \to \perp u$.

In this situation we say that $A$ is cospan representable if this colax functor is horizontally faithful.

8. Some basic cases

(a) For a category $C$ with pullbacks, the weak double category $\text{Span}(C)$ is span representable, in a strict sense: the functor $S: \text{Span}(C) \to \text{Span}(C)$ is an isomorphism, and even the identity for the natural choice of the tabulator of a span, namely its central object. Dually, for every category $C$ with pushouts, $\text{Cospan}(C)$ is ‘strictly’ cospan representable.

(b) On the other hand it is easy to see that $\text{Span}\text{Set}$ is not cospan representable, while $\text{Cospan}\text{Set}$ is not span representable. For the first fact we consider a morphism of spans $\sigma: u \to u$ represented in the left diagram below, where the objects are cardinal sets $(0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\})$

$$\begin{array}{ccc}
1 & \to & 1 \\
\uparrow & & \uparrow \\
2 & \overset{m\sigma}{\to} & 2 \\
\downarrow & & \downarrow \\
1 & \to & 1 \\
\end{array} \quad \begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
2 & \overset{m\sigma}{\to} & 2 \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array} (19)
$$

All the arrows to 1 are determined but the mapping $m\sigma: 2 \to 2$ is arbitrary; the cotabulator pushout is $\perp u = 1$ and $\perp \sigma$ does not determine $\sigma$.

The second counterexample is shown in the right diagram above, where again $m\sigma: 2 \to 2$ is arbitrary, the tabulator pullback is $\top u = 0$ and $\top \sigma$ does not detect $\sigma$.

Similar counterexamples can be given for any category $C$ with finite limits (or colimits) and some object with at least two endomorphisms.

(c) The (strict) double category $A = \text{RelSet}$ of sets, mappings and relations [GP1] has $\text{Hor}_0(A) = \text{Set}$, relations for vertical arrows and (flat) double cells given by an inequality
in the ordered category of relations

\[
\begin{array}{c}
X & \xrightarrow{f} & X' \\
\downarrow{u} & & \downarrow{v} \\
Y & \xrightarrow{g} & Y'
\end{array}
\quad gu \leq vf. \tag{20}
\]

Tabulators and cotabulators exist: \( \top u \subset X \times Y \) is the relation itself, as a subset of \( X \times Y \), while \( \bot u \) is the pushout of the span \( Su = (X \leftarrow \top u \rightarrow Y) \), or of any span representing the relation.

Since \( \mathbb{RelSet} \) is flat it is automatically span and cospan representable. The same holds replacing \( \textbf{Set} \) with any regular category with pushouts.

In the examples below we examine other strict or weak double categories, referring to their definition in [GP1, GP2], briefly reviewed here.

9. Representing profunctors

The weak double category \( \mathcal{C}at \) of categories, functors and profunctors was introduced in [GP1], Section 3.1. Objects are small categories, a horizontal arrow is a functor and a vertical arrow is a profunctor \( u: X \xrightarrow{\cdot} Y \), defined as a functor \( u: X^{\text{op}} \times Y \rightarrow \textbf{Set} \). A cell \( a: (u \xrightarrow{f} v) \) is a natural transformation \( a: u \Rightarrow v(f^{\text{op}} \times g): X^{\text{op}} \times Y \rightarrow \textbf{Set} \). Compositions and comparisons are known or easily defined.

The cotabulator \( \bot u = X +_u Y \) of a profunctor \( u: X \xrightarrow{\cdot} Y \) is the gluing, or collage, of \( X \) and \( Y \) along \( u \), with new maps given by \( (\bot u)(x, y) = u(x, y) \) and no maps ‘backwards’; the composition of the new maps with the old ones is defined by the action of \( u \). The inclusions \( i: X \rightarrow \bot u \) and \( j: Y \rightarrow \bot u \) are obvious, as well as the structural cell \( \iota: (u \xrightarrow{i} e) \)

\[
\iota: u \rightarrow e_{\bot u}(i^{\text{op}} \times j): X^{\text{op}} \times Y \rightarrow \textbf{Set}, \quad \iota(x, y): u(x, y) = \bot u(x, y). \tag{21}
\]

The tabulator \( \top u \) is the category of elements of \( u \), or Grothendieck construction. It has objects \((x, y, \lambda)\) with \( x \in \text{Ob}X, \ y \in \text{Ob}Y, \ \lambda \in u(x, y) \) and maps \((f, g)\) of \( X \times Y \) which form a commutative square in the collage \( X +_u Y \)

\[
(f, g): (x, y, \lambda) \rightarrow (x', y', \lambda'), \quad (f: x \rightarrow x', \ g: y \rightarrow y'), \quad g\lambda = \lambda'f, \quad (u(1_x, g)(\lambda) = u(f, 1_y)(\lambda') \in u(x, y')). \tag{22}
\]

The functors \( p, q \) are obvious, and the structural cell \( \tau = t_u: e_{\top u} \rightarrow u \) is the natural transformation

\[
\tau: e_{\top u} \rightarrow u(p^{\text{op}} \times q): (\top u)^{\text{op}} \times \top u \rightarrow \textbf{Set}, \quad \tau(x, y, \lambda; x', y', \lambda'): \top u(x, y, \lambda; x', y', \lambda') \rightarrow u(x, y'), \quad (f, g) \mapsto g\lambda = \lambda'f. \tag{23}
\]

\( \mathcal{C}at \) is easily seen to be span and cospan representable. Indeed, for a cell \( a: (u \xrightarrow{f} v) \), both the functors \( \top a \) and \( \bot a \) determine every component \( a_{xy}: u(x, y) \rightarrow v(fx, gy) \) of the
natural transformation \( a: u \to v(f^{op} \times g): X^{op} \times Y \to \mathbf{Set} \)

\[
\begin{align*}
\top a: \top u &\to \top v, &\top a(x, y, \lambda) &= (fx, gy, a_{xy}(\lambda)), \\
\bot a: \bot u &\to \bot v, &\bot a(x: x \to y) &= a_{xy}(\lambda): fx \to gy \quad (\lambda \in u(x, y)).
\end{align*}
\]  

(24)

10. Representing adjoints.

We prove now that the double category \( \mathbf{AdjCat} \) of (small) categories, functors and adjunctions, introduced in [GP1], Section 3.5, is also span and cospan representable.

Again \( \text{Hor}_0(\mathbf{AdjCat}) = \mathbf{Cat} \). A vertical arrow is now an ordinary adjunction, conventionally directed as the left adjoint

\[
u = (u_*, u^*, \eta, \varepsilon): X \to Y, \quad (u_*: X \to Y) \dashv (u^*: Y \to X), \quad \eta: 1_X \to u_*u^*, \quad \varepsilon: u_*u^* \to 1_Y.
\]  

(25)

A double cell \( a = (a_*, a^*): u \to v \) is a pair of mate natural transformations, each of them determining the other via the units and counits of the two adjunctions

\[
a_*: v_*f \to gu_*, \quad a^*: fu^* \to v^*g, \quad a_* = (fu^* \to v^*v_ufu^* \to v^*gu_*u^* \to v^*g), \quad a^* = (v_*f \to v_*fu^*u_* \to v_*v^*gu_* \to gu_*)
\]  

(26)

(a) In \( \mathbf{AdjCat} \) the tabulator \( \top u \) of an adjunction \( u = (u_*, u^*): X \to Y \) is the ‘graph’ of the adjunction, namely the following comma category, equipped with the comma-projections \( p, q \) and an obvious cell \( \tau = t_u: (e \circ q \circ u) \)

\[
\begin{align*}
\top u &\cong (X \downarrow u^*), & (x, y; c: u_*x \to y) &\leftrightarrow (x, y; c': x \to u^*y), \\
p: \top u &\to X, & q: \top u &\to Y, \\
\tau_*(x, y; c) &\circ u_* \to Y, & \tau_*(x; y, c) &\circ c: u_*x \to y.
\end{align*}
\]  

(27)

The tabulator of a cell \( a: (u^* f v) \), with components \( a_* x: v_* fx \to gu_* x \), is the following functor

\[
\top a: \top u \to \top v, \quad \top a(x, y; c: u_*x \to y) = (fx, gy; g(c)a_* x: v_* fx \to gy).
\]  

(28)

This proves that \( \mathbf{AdjCat} \) is span representable: in fact the component \( a_* x: v_* fx \to gu_* x \) is determined by \( \top a(x, u_* x; 1: u_* x \to u_* x) = (fx, gu_* x; a_* x: v_* fx \to gu_* x) \).

(b) In \( \mathbf{AdjCat} \) the cotabulator \( C = \bot u = X +_u Y \) is the category consisting of the disjoint union \( X + Y \), together with new maps \( c = (x, y; c: u_*x \to y)^{\sim} \in C(x, y) \) from objects of \( X \) to objects of \( Y \) that are ‘represented’ by objects \( (x, y; c: u_*x \to y) \) of \( \top u = (u_* \downarrow Y) \): the composition of the new maps with old maps \( \varphi \in X(x', x), \psi \in Y(y, y') \) is defined in the obvious way

\[
\psi_\varphi = \lambda x': (x', y'; (x, c) \circ u_* (\varphi): u_* x' \to u_* x \to y \to y').
\]  

(29)
The universal cell $ι: i \rightarrow ju: X \rightarrow ⊥u$ is given by $ιx = (1u_ux)^{−} ∈ C(x, u_ux)$. The cotabulator of a cell $a: (u f g v)$, with components $a_ux: v_uxf x → gu_ux$, works as $f$ and $g$ on the old objects and arrows, as $⊤a$ on the new arrows

$$\perp a: ⊥u → ⊥v, \quad \perp a(x, y; h: u_ux → y)^{−} = (fx, gy; g(h).a_ux: v_uxf x → gy)^{−}. \quad (30)$$

This determines $a_ux$ as above.

11. Representing $\mathbb{D}bl$

The strict double category $\mathbb{D}bl$ of weak double categories, lax functors and colax functors is a crucial structure, on which the theory of double adjoints is based. We refer the reader to its introduction in [GP2], Section 2, where the non-obvious point of double cells is dealt with.

We prove now that $\mathbb{D}bl$ is span representable, horizontally and vertically.

(a) First, every colax functor $U: A • → B$ has a horizontal tabulator $(T, P, Q, τ)$.

The weak double category $T = U ↓ B$ is a ‘one-sided’ double comma ([GP2], Section 2.5), with strict projections $P$ and $Q$, which can be used as horizontal or vertical arrows. Below the cell $η$ is simply represented by the horizontal transformation $1Q: Q → Q$ and the tabulator cell $τ = t_U$ is linked to the comma-cell $π$ by the unit $η$ and counit $ε$ of the companionship of $Q$ with ‘itself’ (see [GP2])

$$\tau = (η | π), \quad π = (ε). \quad (31)$$

To be more explicit, the tabulator $T$ has objects

$$(A, B, b: UA → B), \quad (32)$$

with $A$ in $A$ and $b$ horizontal in $B$. A horizontal arrow of $T$

$$(a, b): (A_1, B_1, b_1) → (A_2, B_2, b_2), \quad (33)$$

‘is’ a commutative square in $\text{Hor}_0B$, as in the upper square of diagram (35), below (where the slanting direction must be viewed as horizontal). A vertical arrow of $T$

$$(u, v, w): (A_1, B_1, b_1) → (A_3, B_3, b_3), \quad (34)$$

‘is’ a double cell in $B$, as in the left square of diagram (35). A double cell $(β, β′)$ of $T$ forms a commutative diagram of double cells of $B$

$$(β, β′): ((u, v, ω) (a, b) (u′, v′, ω′)), \quad (ω | β′) = (β | ω), \quad (35)$$
The composition laws of $T$ are obvious, as well as the (strict) double functors $P, Q$. The double cell $\tau$ has components

$$\tau(A, B, b) = b: UA \to B,$$

$$\tau(u, v, \omega) = \omega: Uu \to v.$$  \tag{36}$$

Its universal property follows trivially from that of the double comma, in [GP2], Theorem 2.6(a).

(b) We have thus a span representation

$$S: \mathbb{dbl} \to \text{Span}(\text{Lxdbl}),$$  \tag{37}$$

where $\text{Lxdbl} = \text{Hor}_0\mathbb{dbl}$ is the category of weak double categories and lax functors. (Note that, even though the projections $P, Q$ of the double comma $T$ are strict double functors, a cell $\varphi: (U,F,G,V)$ in $\mathbb{dbl}$ gives a lax functor $\top\varphi: \top U \to \top V$.)

To prove that $\mathbb{dbl}$ is horizontally span representable, we use the \textit{vertical} universal span property of the double comma $T = U \downarrow B$, in [GP2], Theorem 2.6(b), and deduce the existence of a colax functor $W: A \to T$ and a cell $\xi$ such that:

$$A \xrightarrow{W} \quad \xrightarrow{\xi} \quad A \xrightarrow{1} \quad (QW = U).$$  \tag{38}$$

Now a cell $\varphi: (U,F,G,V)$ in $\mathbb{dbl}$ can be recovered from the lax functor $\top\varphi: \top U \to \top V$ as follows

$$\varphi = (1_U | \varphi) = (\xi \otimes \pi | e_F \otimes \varphi) = (\xi \otimes t_U \otimes \varepsilon | e_F \otimes \varphi \otimes e_G)$$

$$= (\xi | e_F) \otimes (t_U | \varphi) \otimes (\varepsilon | e_G) = (\xi | e_F) \otimes (\top\varphi | t_V) \otimes (\varepsilon | e_G).$$

(c) Transpose duality leaves $\mathbb{dbl}$ invariant up to isomorphism: sending an object $A$ to the horizontal opposite $A^h$ and transposing double cells we have an isomorphism $\mathbb{dbl} \to \mathbb{dbl}^t$. Therefore $\mathbb{dbl}$ is also \textit{vertically span representable}, which means that $\mathbb{dbl}^t$ is span representable by a lax functor

$$S': \mathbb{dbl}^t \to \text{Span}(\text{Cxdbl})$$  \tag{39}$$

where $\text{Cxdbl} = \text{Hor}_0\mathbb{dbl}^t = \text{Ver}_0\mathbb{dbl}$. 

\[\text{Diagram:} \]

1. $Ua \rightarrow \bullet \rightarrow Uu$ with $a, \omega, b$.
2. $Uu \rightarrow \bullet \rightarrow Ua$ with $u, v, \beta$.
3. $Ua \rightarrow \bullet \rightarrow Uu$ with $a', u'$, $\omega'$, $v'$.
The latter sends a lax functor $F : A \to B$ to the span $S'(F) = (A \leftarrow T' \to B)$ associated to its vertical tabulator $(T, P, Q, \tau)$, where the weak double category $T = B \downarrow F$ has objects $(A, B, b : B \to FA)$, and the cell $\tau$ is vertically universal

\[
\begin{array}{c}
\begin{array}{c}
\mathcal{T} \\
\mathcal{A}
\end{array}
\end{array}
\quad \xrightarrow{\begin{array}{c}
1 \\
\tau
\end{array}}
\quad \begin{array}{c}
\mathcal{T} \\
\mathcal{B}
\end{array}
\]

12. Theorem (Representing quintets)

The 2-category $\mathcal{C}$ is 2-complete if and only if the associated double category $\mathcal{Q}\mathcal{C}$ of quintets has all double limits. In this case the double category $\mathcal{Q}\mathcal{C}$ is span representable.

PROOF. Let us recall that the double category $\mathcal{Q}\mathcal{C}$ of quintets (introduced by C. Ehresmann) has for horizontal and vertical maps the morphisms of $\mathcal{C}$, while its double cells are defined by 2-cells of $\mathcal{C}$

\[
\begin{array}{c}
\begin{array}{c}
X \\
Y
\end{array}
\end{array}
\xrightarrow{\begin{array}{c}
f \\
\varphi
\end{array}}
\quad \begin{array}{c}
X' \\
Y'
\end{array}
\quad \varphi : vf \to gu : X \to Y'.
\]

(41)

It is known that $\mathcal{C}$ is 2-complete if and only if it has 2-products, 2-equalisers and cotensors by the arrow-category $\mathcal{2}$ [St]. First, it is easy to see that 2-products (resp. 2-equalisers) in $\mathcal{C}$ are ‘the same’ as double products (resp. double equalisers) in $\mathcal{Q}\mathcal{C}$. Second, the cotensor $\mathcal{2} * X$ can be obtained as the tabulator of the vertical identity of $X$: they are defined by the same universal property.

Conversely, if the $\mathcal{C}$-morphism $u : X \to Y$ is viewed as vertical in $\mathcal{Q}\mathcal{C}$, its tabulator $(\top u; p, q; \tau)$ can be constructed as the following inserter $(\top u; i, \tau)$

\[
\begin{array}{c}
\begin{array}{c}
\top u \\
\tau
\end{array}
\end{array}
\quad \begin{array}{c}
X \times Y \\
\mathcal{Y}
\end{array}
\xrightarrow{\begin{array}{c}
up' \\
p'
\end{array}}
\quad \begin{array}{c}
Y \\
\mathcal{Y}'
\end{array}
\quad \tau : up'i \to p''i : \top u \to Y,
\]

(42)

letting $p = p''i : \top u \to X$, $q = p''i : \top u \to Y$ and viewing $\tau$ as a double cell with boundary $(1_p, u)$.

If $\mathcal{C}$ is 2-complete, $\mathcal{Q}\mathcal{C}$ is span representable because the lax span representation $S : \mathcal{Q}\mathcal{C} \to \text{Span}(\mathcal{C})$ operates on a double cell $a : vf \to gu$ of $\mathcal{Q}\mathcal{C}$ producing a morphism of spans $Sa : Su \to Sv$ whose central map $\top a : \top u \to \top v$ is defined as follows

\[
\begin{array}{c}
t_u : up'i \to p''i : \top u \to Y, \\
\quad t_v : vq'j \to q''j : \top v \to Y', \\
\quad j.\top a = (f \times g)i, \\
\quad t_v.\top a = gt_u.ap'i : vfp'i \to gp''i.
\end{array}
\]

(43)
\[
\begin{array}{c}
\top u \xrightarrow{i} X \times Y \xrightarrow{up'} Y \\
\top a \xrightarrow{fg} X' \times Y' \xrightarrow{q'} Y'
\end{array}
\]

Now \(f\) and \(g\) are determined as the vertical faces of the morphism \(Sa\). To recover the 2-cell \(a\): \(vf \rightarrow gu\) of \(C\) from the morphism \(\top a\), one uses the map \(h: X \rightarrow \top u\) determined by the conditions \(ih = (1, u): X \rightarrow X \times Y\) and \(t_u.h = 1_u\), so that

\[
t_u.\top a.h = (gt_u.ap'i)h = gt_u.h.ap'ih = ap'(1, u) = a.
\]

13. Splitting tabulators

In order to ‘explain’ how so many double categories are span representable, we observe that the proof for the non-obvious cases above follows a pattern of the following type (as in Section 12), or a vertical version of the same (as in Section 11). However the argument is rather complicated, and - in the examples above - we preferred to give a direct proof, following this guideline.

We are in a weak double category \(\mathcal{A}\) with tabulators, and the category \(\mathcal{C} = \text{Hor}_0(\mathcal{A})\) has pullbacks. In order that \(\mathcal{A}\) be span representable it is sufficient that, for every vertical arrow \(u\), there exist two cells \(s_u\) and \(\varepsilon\) satisfying the following condition:

\[
\begin{array}{c}
X \xrightarrow{h} \top u \xrightarrow{\varepsilon} X \xrightarrow{\rho u} X \\
\downarrow s_u \searrow t_u \downarrow u \\
Y \xrightarrow{\varepsilon} Y \xrightarrow{\rho u} Y
\end{array}
\]

\[
\begin{array}{c}
1_u = (\top a) = (s_u | t_v | \rho u | a) = (s_u | t_u | a | \rho v) = (s_u | t_u | a | \rho v) = (s_u | \top a | t_v | \rho v)
\end{array}
\]

(Typically, \(q\) is the vertical companion of \(q\) and \(\varepsilon = \varepsilon_q\) its counit, but this is not needed in the proof. In the strict case \(\rho u\) is trivial.)

In fact one can recover a cell \(a\): \((u f v)\) from \(\top a\) (and \(Sa\)), as follows

\[
a = (s_u | t_u | \varepsilon | \rho u | a) = (s_u | t_u | a | \varepsilon | \rho v) = (s_u | t_u | a | \varepsilon | \rho v) = (s_u | \top a | t_v | \rho v).
\]

References


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