# FILTERED DOUBLE LIMITS

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- IN UNIV. ALG. — FILT COLIM CENTRAL
- FILT COLIM COMMUTE W. FINTE LIM IN SET
- GENERALIZE TO 2-CATS (2-DIM. ALG.)
- LIMITS IN 2-CATS ???
- $\underline{V}$-CAT : WEIGHTED COLIMS

  $\Phi : \underline{I} \to \underline{A}$, $W : \underline{I}^{op} \to \underline{V}$

  $$\varinjlim_W \Phi = \int^I W(I) \otimes \Phi(I)$$

- $\underline{V} = \underline{SET}$, $El(W) \longrightarrow \underline{I} \longrightarrow \underline{A}$
- $\underline{V} = \underline{CAT}$, $El(W) = ??$

  THE ONE THAT WORKS IS A **DOUBLE CATEGORY!**

- DOUBLE LIM $\leftrightarrow$ WEIGHTED LIM, BUT...

# FILTERED COLIMITS

$\underline{J}$ **FILTERED** IFF

(1) $\underline{J} \neq \emptyset$

(2) $\forall A, B \; \exists \; A \to C \leftarrow B$ (3) $\forall A \rightrightarrows B \; \exists \; A \rightrightarrows B \to C$

**THM**: TFAE:

(1) $\underline{J}$ $\varinjlim$ COMMUTE WITH ALL FINITE $\varprojlim$ IN $\underline{\text{SET}}$

(2) EVERY FINITE DIAG IN $\underline{J}$ HAS COCONE

(3) $\underline{J}$ FILTERED

**PF**: (2) $\Leftrightarrow$ (3) ✓

(2) $\Rightarrow$ (1) CALCULATION

(1) $\Rightarrow$ (2)

$D : \underset{\text{FIN}}{\underline{I}} \longrightarrow \underline{J}$

$$\Phi : \underline{I}^{op} \times \underline{J} \xrightarrow{\Phi^{op} \times \underline{J}} \underline{J}^{op} \times \underline{J} \xrightarrow{\text{HOM}} \underline{\text{SET}}$$

$$\varinjlim_{J} \varprojlim_{I} \Phi(I,J) \longrightarrow \varprojlim_{I} \varinjlim_{J} \Phi(I,J)$$

EQUIV CL $\qquad\qquad\qquad\qquad$ "

OF COCONES $\qquad\qquad\qquad$ 1

# HOM FOR DOUBLE CATS

· DOUBLE CAT = CAT IN $\underline{CAT}$

$$\mathbb{I} : \underline{I}_2 \rightrightarrows \underline{I}_1 \rightleftarrows \underline{I}_0$$

HOR + DOUB　　OBJ + VERT

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "w"] \\ \bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \end{tikzcd}$$

HOR + VERT COMPOSITION

HOM : $\mathbb{I}^{op} \times \mathbb{I} \longrightarrow CAT$ ?

$$(A, B) \longmapsto \begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

HOR FUNCTORIAL

$$\begin{tikzcd} A \arrow[d, "v"'] & B \arrow[d, "w"] \\ \bar{A} & \bar{B} \end{tikzcd} \longmapsto \left\{ \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "w"] \\ \bar{A} \arrow[r, "g"'] & \bar{B} \end{tikzcd} \right\}$$

SET [illegible]

$$Hom(A,B)^{op} \times Hom(\bar{A}, \bar{B}) \longrightarrow \underline{SET}$$

A PROFUNCTOR

$$Hom(v,w) : Hom(\bar{A}, \bar{B}) \longrightarrow\!\!\!\!\!\!\circ\;\; Hom(A,B)$$

# THE SECRET DOUBLE LIFE OF CAT

A **PROFUNCTOR** $P : \underline{A} \mathrel{-\!\!\bullet\!\!\to} \underline{B}$ IS

$P : \underline{B}^{op} \times \underline{A} \to \underline{SET} \quad (\underline{A} \to \underline{SET}^{\underline{B}^{op}})$

$Q : \underline{B} \mathrel{-\!\!\bullet\!\!\to} \underline{C}$

$$Q \otimes P(C,A) = \int^{B} Q(C,B) \times P(B,A)$$

$$I_A = \mathrm{HOM}_{\underline{A}} : \underline{A}^{op} \times \underline{A} \to \underline{SET}.$$

$F : \underline{A} \to \underline{B}, \quad F_*(B,A) = \underline{B}(B, FA)$

$$F^*(A,B) = \underline{B}(FA, B)$$

NOTATION $\quad B \mathrel{\overset{x}{\underset{P}{-\!\!\bullet\!\!\to}}} A \quad$ IS $\quad x \in P(B,A).$

## $\mathbb{P}$CAT  WEAK DOUBLE CAT

$$\begin{tikzcd}
\underline{A} \arrow[r, "F"] \arrow[d, "P"', "\bullet" marking] \arrow[dr, phantom, "\overset{t}{\Rightarrow}"] & \underline{B} \arrow[d, "\bullet" marking, "Q"] \\
\underline{\bar{A}} \arrow[r, "F'"'] & \underline{\bar{B}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{A} \arrow[d, "a"', "\bullet" marking] \arrow[dr, phantom, "\overset{t}{\longmapsto}"] & F\bar{A} \arrow[d, "\bullet" marking, "ta"] \\
A & F'A
\end{tikzcd}$$

$\mathrm{HOM} : \mathbb{I}^{coop} \times \mathbb{I}^{co} \to \mathbb{P}\mathrm{CAT}$ LAX NORMAL!

# THE GROTHENDIECK SEMIDIRECT PRODUCT

$\Phi : \mathbb{I}^{op} \longrightarrow \mathbb{P}\mathrm{CAT}$ LAX NORMAL

$\mathbb{I}\Phi$ DOUBLE CAT OF "ELEMENTS"

$$\begin{tikzcd}
(I,A) \arrow[r, "{(i,A')}"] \arrow[d, "{(v,x)}"'] \arrow[dr, phantom, "{\Downarrow (\alpha,x')}"] & (I',A') \arrow[d, "{(v',x')}"] \\
(\bar{I},\bar{A}) \arrow[r, "{(\bar{i},\bar{A}')}"'] & (\bar{I}',\bar{A}')
\end{tikzcd}$$

$A \in \Phi I$

$\Phi(i)(A') = A$

$\bar{A} \xrightarrow[\Phi(v)]{x} A$

$\Phi(\alpha)(x') = x$

- $\mathbb{I}\Phi \xrightarrow{\Pi} \mathbb{I}$ DOUBLE DISC FIBRATION

<u>PROP</u>: $\Phi$ REP $\Leftrightarrow$ $\mathbb{I}\Phi$ HAS TERMINAL.

<u>REPRESENTABLE</u>: HORIZ ISO TO $\mathrm{Hom}(-,A)$

<u>TERMINAL OBJ</u>:

$$\forall\ \begin{tikzcd}
A \arrow[dr, "\exists!"] \arrow[d, "v"'] \arrow[dr, phantom, shift right=3, "\Downarrow \exists!"] & \\
\bar{A} \arrow[r, "\exists!"'] & T
\end{tikzcd}$$

# DOUBLE LIM & COLIM IN $\mathbb{P}$CAT

$\Phi : \mathbb{I}^{op} \longrightarrow \mathbb{P}\mathrm{Cat}$ LAX NORMAL

$$
\begin{tikzcd}
 & \Phi I' \arrow[d, "\Phi i"] \\
\underleftarrow{\mathrm{Lim}}\,\Phi \arrow[ur, "\lambda I'"] \arrow[r, "\lambda I"] \arrow[dr, "\lambda \bar{I}"'] \arrow[r, phantom, shift right=4ex, "\Downarrow \lambda v"] & \Phi I \arrow[d, "\Phi v", "\bullet"'] \\
 & \Phi \bar{I}
\end{tikzcd}
$$

HORIZ NAT
VERT COMPOSES
&
UNIVERSAL

IN $\mathbb{P}$CAT: $\underleftarrow{\mathrm{Lim}}\,\Phi$ = SECTIONS OF $\int\Phi \xrightarrow{\pi} \mathbb{I}$

$$\underrightarrow{\mathrm{Lim}}\,\Phi = \pi_0 \textstyle\int\Phi \twoheadleftarrow \textstyle\int\Phi_0 \leftleftarrows \textstyle\int\Phi_1$$

ON TRANSFORMATIONS (VERT) $\Phi \xrightarrow{u} \bar{\Phi}$

DEFINED BY UNIV PROPERTY

ONLY LAX NORMAL!

# FILTERED DOUBLE COLIMITS

$\Phi : \mathbb{I}^{op} \times \mathbb{J} \longrightarrow \mathbb{P}\mathrm{Cat}$ LAX NORMAL

<u>NOTE</u>: $\varprojlim_{I} \Phi(I,J)$ NOT FUNCTURIAL IN $J$ EVEN IF $\Phi$. THUS LAX!

$$\varinjlim_{J} \varprojlim_{I} \Phi(I,J) \xrightarrow{\varphi} \varprojlim_{I} \varinjlim_{J} \Phi(I,J)$$

<u>THM</u>: GIVEN $\mathbb{J}$, TFAE:

(1) $\forall$ FINITE $\mathbb{I}$, $\varphi$ ISO.

(2) EVERY FINITE DIAGRAM IN $\mathbb{J}$ HAS A COCONE

(3) (i) $\mathbb{J} \neq \emptyset$

(ii) $\forall A, B \; \exists$
$$\begin{tikzcd} A \arrow[dr] & \\ & C \\ B \arrow[ur] & \end{tikzcd}$$

(iii) $\forall A \rightrightarrows B \; \exists A \rightrightarrows B \to C$

(iv) $\forall$
$$\begin{tikzcd} A \\ \bar{A} \arrow[u, "\bullet"] \end{tikzcd} \quad \exists \quad \begin{tikzcd} A \arrow[dr] & \\ & B \\ \bar{A} \arrow[uu, "\bullet"] \arrow[ur] \arrow[u, phantom, "\Downarrow"] & \end{tikzcd}$$

(v) $\forall$
$$\begin{tikzcd} A \arrow[dr] & \\ & B \\ \bar{A} \arrow[uu, "\bullet"] \arrow[ur] \arrow[u, phantom, "\Rrightarrow"] & \end{tikzcd}$$

$\exists$
$$\begin{tikzcd} A \arrow[dr] & & \\ & B \arrow[r] & C \\ \bar{A} \arrow[uu, "\bullet"] \arrow[ur] \arrow[u, phantom, "\Rrightarrow"] & & \end{tikzcd}$$

$\boxed{\mathbb{J} \text{ FILTERED}}$