

# FILTERED DOUBLE LIMITS

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- IN UNIV. ALG. — FILT COLIM CENTRAL
- FILT COLIM COMMUTE W. FINTE LIM IN SET
- GENERALIZE TO 2-CATS (2-DIM. ALG.)
- LIMITS IN 2-CATS ???
- $\underline{V}$ -CAT: WEIGHTED COLIMS  
 $\Phi: \underline{I} \rightarrow \underline{A}$ ,  $W: \underline{I}^{\circ} \rightarrow \underline{V}$   
 $\lim_{\rightarrow W} \Phi = \int^{\underline{I}} W(I) \otimes \Phi(I)$
- $\underline{V} = \underline{SET}$ ,  $El(W) \rightarrow \underline{I} \rightarrow \underline{A}$
- $\underline{V} = \underline{CAT}$ ,  $El(W) = ??$   
THE ONE THAT WORKS IS A  
**DOUBLE CATEGORY!**
- DOUBLE LIM  $\leftrightarrow$  WEIGHTED LIM, BUT...

# FILTERED COLIMITS

1.5

J FILTERED IFF

(1)  $\underline{J} \neq \emptyset$

(2)  $\forall A, B \exists A \rightrightarrows_B C$  (3)  $\forall A \exists B \exists A \rightrightarrows B \rightarrow C$

THM : TFAE :

(1) J  $\varinjlim$  COMMUTE WITH ALL FINITE  $\varprojlim$  IN SET

(2) EVERY FINITE DIAG IN J HAS COCONE

(3) J FILTERED

PF: (2)  $\Leftrightarrow$  (3)  $\checkmark$

(2)  $\Rightarrow$  (1) CALCULATION

(1)  $\Rightarrow$  (2)

$$D: \underline{I} \rightarrow \underline{J}$$

$$\Phi: \underline{I}^{\text{FN}} \times \underline{J} \xrightarrow{\Phi \times \text{J}} \underline{J}^{\text{FN}} \times \underline{J} \xrightarrow{\text{HOM}} \underline{\text{SET}}$$

$$\varinjlim_{\underline{J}} \varprojlim_{\underline{I}} \Phi(I, J) \rightarrow \varprojlim_{\underline{I}} \varinjlim_{\underline{J}} \Phi(I, J)$$

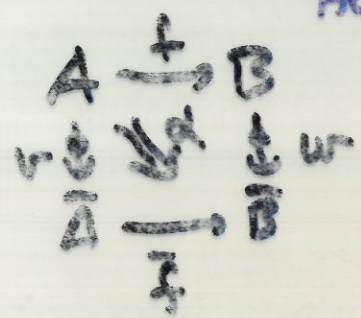
EQUIV CL

# HOM FOR DOUBLE CATS

DOUBLE CAT = CAT IN CAT

$$\text{II: } \underline{\text{I}}_2 \rightleftarrows \underline{\text{I}}_1 \rightleftarrows \underline{\text{I}}_0$$

↑ HOR + VERT      ↑ OBJ + VERT  
↑ HOR + DOUB



HOR + VERT  
COMPOSITION

$$\text{HOM: } \text{II}^{\text{OP}} \times \text{II} \longrightarrow \text{CAT} \quad ?$$

$$(A, B) \longmapsto A \overset{\circlearrowright}{\Downarrow} B$$

HOR FUNCTORIAL

$$\begin{matrix} A & B \\ v \downarrow & \downarrow w \\ \bar{A} & \bar{B} \end{matrix} \longmapsto \left\{ \begin{matrix} A \xrightarrow{f} B \\ v \downarrow \downarrow w \\ \bar{A} \xrightarrow{\bar{f}} \bar{B} \end{matrix} \right\}$$

SET OF...

$$\text{HOM}(A, B)^{\text{OP}} \times \text{HOM}(\bar{A}, \bar{B}) \longrightarrow \text{SET}$$

## A PROFUNCTOR

$$\text{HOM}(v, w): \text{HOM}(\bar{A}, \bar{B}) \longrightarrow \text{HOM}(A, B)$$

# THE SECRET DOUBLE LIFE OF CAT

3.

A PROFUNCTOR  $P: \underline{A} \rightarrow \underline{B}$  is

$$P: \underline{B}^{op} \times \underline{A} \rightarrow \underline{SET} \quad (\underline{A} \rightarrow \underline{SET}^{\underline{B}^{op}})$$

$$Q: \underline{B} \rightarrow \underline{C}$$

$$Q \otimes P(C, A) = \int^B Q(C, B) \times P(B, A)$$

$$I_A = \text{HOM}_A: \underline{A}^{op} \times \underline{A} \rightarrow \underline{SET}.$$

$$F: \underline{A} \rightarrow \underline{B}, \quad F_* (B, A) = \underline{B}(B, FA)$$

$$F^* (A, B) = \underline{B}(FA, B)$$

NOTATION  $B \overset{x}{\underset{P}{\dashrightarrow}} A$  is  $x \in P(B, A)$ .

## PCAT WEAK DOUBLE CAT

$$\begin{array}{ccc}
 \underline{A} & \xrightarrow{F} & \underline{B} & & \underline{\bar{A}} & & \underline{F\bar{A}} \\
 \underline{P} \downarrow & \xRightarrow{\tau} & \downarrow \underline{Q} & & a \downarrow & \xrightarrow{\tau} & \downarrow \tau a \\
 \underline{\bar{A}} & \xrightarrow{F'} & \underline{\bar{B}} & & \underline{A} & & \underline{F'A}
 \end{array}$$

$$\text{HOM}: \underline{II}^{coop} \times \underline{II}^{co} \rightarrow \text{PCAT}$$

LAX  
NORMAL!

# THE GROTHENDIECK SEMIDIRECT PRODUCT

$$\Phi: \mathbb{I}^{\text{op}} \rightarrow \text{PCAT} \quad \text{LAX NORMAL}$$

$\mathbb{I}\Phi$  DOUBLE CAT OF "ELEMENTS"

$$\begin{array}{ccc}
 (I, A) \xrightarrow{(\iota, A')} (I', A') & & A \in \Phi I \\
 (\nu, x) \downarrow \Downarrow (\alpha, x') \downarrow (\nu', x') & & \Phi(\iota)(A') = A \\
 (I, \bar{A}) \xrightarrow{(\bar{\iota}, \bar{A}')} (I', \bar{A}') & & \bar{A} \xrightarrow[\Phi(\nu)]{x} A \\
 & & \Phi(\alpha)(x') = x
 \end{array}$$

•  $\mathbb{I}\Phi \xrightarrow{\Pi} \mathbb{I}$  DOUBLE DISC FIBRATION

PROP:  $\Phi \text{ REP} \Leftrightarrow \mathbb{I}\Phi \text{ HAS TERMINAL}$ .

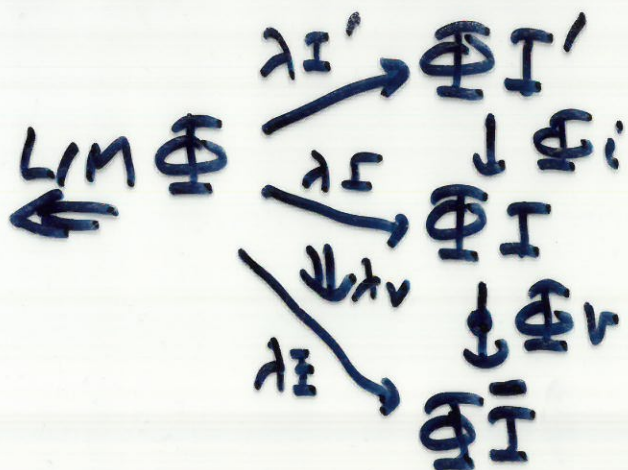
REPRESENTABLE: HORIZ ISO TO  $\text{HOM}(-, A)$

TERMINAL OBJ:



DOUBLE LIM & COLIM IN PCAT

$\Phi: \mathbb{I}^{op} \rightarrow \mathbb{P}CAT$  LAX NORMAL



HORIZ NAT  
VERT COMPOSES  
&  
UNIVERSAL

IN PCAT:  $\lim \Phi =$  SECTIONS OF  $\Gamma \Phi \xrightarrow{\pi} \mathbb{I}$

$$\lim \Phi = \pi_0 \Gamma \Phi \leftarrow \Gamma \Phi_0 \leftarrow \Gamma \Phi_1$$

ON TRANSFORMATIONS (VERT)  $\Phi \xrightarrow{\gamma} \bar{\Phi}$   
DEFINED BY UNIV PROPERTY  
ONLY LAX NORMAL!

# FILTERED DOUBLE COLIMITS

$$\Phi: \mathbb{I}^{op} \times \mathbb{J} \rightarrow \mathcal{PCAT} \quad \text{LAX NORMAL}$$

NOTE:  $\varprojlim_{\mathbb{I}} \Phi(I, J)$  NOT FUNCTORIAL IN  $J$  EVEN IF  $\Phi$ . THUS LAX!

$$\varinjlim_{\mathbb{J}} \varprojlim_{\mathbb{I}} \Phi(I, J) \xrightarrow{\varphi} \varprojlim_{\mathbb{I}} \varinjlim_{\mathbb{J}} \Phi(I, J)$$

THM: GIVEN  $\mathbb{J}$ , TFAE:

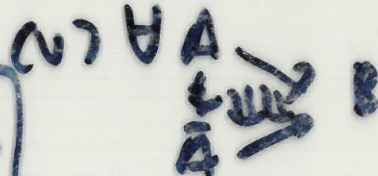
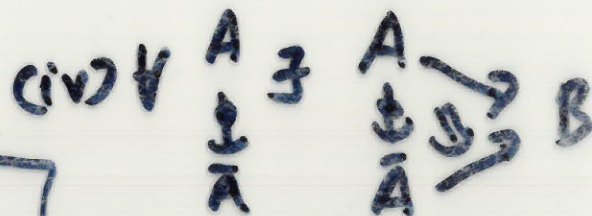
(1)  $\forall$  FINITE  $\mathbb{I}$ ,  $\varphi$  ISO.

(2) EVERY FINITE DIAGRAM IN  $\mathbb{J}$  HAS A COCONE

(3) (i)  $\mathbb{J} \neq \emptyset$

(ii)  $\forall A, B \in \mathbb{J} \exists A \rightarrow B$

(iii)  $\forall A \ni B \exists A \rightarrow B \rightarrow C$



**$\mathbb{J}$  FILTERED**