INTERCATEGORIES: A FRAMEWORK FOR THREE-DIMENSIONAL CATEGORY THEORY

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ABSTRACT. We show how the notion of intercategory encompasses a wide variety of three-dimensional structures from the literature, notably duoidal categories, monoidal double categories, cubical bicategories, double bicategories and Gray categories. Variations on the notion of span provide further examples of interest, an important one being the intercategory of sets. We consider the three kinds of morphism of intercategory as well as the cells binding them with applications to the above structures. In particular hom functors are studied.

Introduction

In this paper we show how intercategories, introduced in [13], provide a conceptual framework for three-dimensional category theory. Many notions of three-dimensional category already appear in the literature each with its own use, and no doubt many more will appear as the theory develops. Some of the more established ones which we discuss here are duoidal categories [1, 3, 4], monoidal double categories [20], cubical bicategories [7], Verity's double bicategories [21] and Gray [8]. As will be seen below, these can all be considered as special intercategories. The range and variety of examples is a testament to the unifying role intercategories can play. Not only do they provide an effective organization and unification of a large number of three-dimensional structures from the literature, but by putting these in a common setting it is possible to consider morphisms between them and study how they relate to each other. The more encompassing context often suggests useful generalizations which are more natural and actually come up in practice.

There are many ways of looking at intercategories each with its own intuition, complementing and augmenting the others. At the most basic level an intercategory is a laxified triple category, having three kinds of arrows, three kinds of cells relating these pairwise, and cubes relating these. We haven't strived for the most general laxity possible, but rather a specific choice, informed by the examples. This will become apparent as the paper progresses.

Intercategories are a natural extension of double categories to the next dimension. Just as double categories can be conceptually understood as two categories with the same objects, intercategories can be thought of as two double categories with a common

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horizontal category. Of course, the two double categories are related in more than this shared horizontal structure. There are three-dimensional cells, called cubes, and this is where the laxity comes in. This point of view is exploited in Section 6 where various double categories of spans or cospans interact.

A different connection to double categories, explored in Section 4, is that intercategories may be thought of as double categories whose horizontal *and* vertical compositions are weak, with an added transversal direction which is strict and used to express the coherence conditions. Weak double categories are strict in the horizontal direction; yet there are many cases where both compositions are weak, e.g. quintets in a bicategory. Trying to formalize this as a 2-dimensional structure leads to a vicious circle.

One of the motivating examples for our own definition of intercategory was 2-monoidal or duoidal categories [1, 3, 4]. These are categories with two monoidal structures in which one of the tensors and its unit are monoidal with respect to the other. It is tempting to try to place this in the context of double categories, i.e. a double category with one object. Even if we allow weak composition in both directions, this doesn't work. It is only at the level of intercategories that it does. So we can think of an intercategory as a "duoidal category with several objects". This point of view is studied in Section 2.

The main reason for defining 2-monoidal categories in [1] was to study their morphisms and thus put some order in the large number of structures arising in Hopf algebra theory. There are three types of morphism corresponding to our lax/lax, colax/lax, and colax/colax functors. The "intercategory as two double categories" point of view would suggest four types, but the lax/colax (lax in the horizontal direction and colax in the vertical) don't come up. In fact they don't make sense! Having three types of morphism suggests that there is a triple category of intercategories, which is indeed the case. There are appropriate 2-cells and commutative cubes all fitting together nicely in a strict triple category. This is perhaps the main theorem of [13].

This suggests a further point of view, namely that the triple category of intercategories is a universe for doing higher category theory. Thus in Section 7 we study "hom functors" for intercategories and single out one particular intercategory where they take their values, which we believe can rightfully be called **Set**, the intercategory of sets. Much remains to be done here.

Conspicuously lacking in our examples are tricategories [8], which were a major stepping stone to the world of higher category theory. Strict 3-categories can be considered as strict triple categories in which the transversal and horizontal arrows are identities, as well as the horizontal and vertical cells, so they are special intercategories. There is a more symmetric way of considering a 3-category as an intercategory, a higher dimensional version of quintets, in which the three-dimensional cells are cubes containing a 3-cell and whose faces are compatible quintets. But none of this works for tricategories where the composition is only associative up to *equivalence*.

In fact our work on intercategories started when we realized that there seem to be no naturally occurring non-trivial examples of genuinely weak triple categories. What is meant is some natural example where there are three types of morphisms, the first strictly associative, the second associative up to coherent isomorphism and the third only associative up to equivalence. In a sense intercategories are a competing notion. They are simpler than tricategories because the associativity and unit constraints are isomorphisms, and many tricategories come from naturally occurring intercategories in a way to be explained shortly. In these cases the fact that the coherence conditions are equivalences rather than isomorphisms is a result of requiring globularity.

We start with Shulman's monoidal double categories [20], which we explore in Section 3. His idea is that, whereas it is a lot of work to check that something is a monoidal bicategory, let alone a symmetric one, it is comparatively easy to check the conditions for a (symmetric) monoidal double category. Every double category has a vertical bicategory gotten by discarding the non-identity horizontal arrows. But the constraint isomorphisms for a monoidal double category are horizontal isomorphisms, so a replacement has to be found. In reasonable double categories, horizontal isomorphisms have companions, and these are always equivalences. What Shulman proves in [20] is that this gives a (symmetric) monoidal bicategory. He claims that most monoidal bicategories that come up in practice arise in this way, and goes on to say that in these cases it's usually better to work with the double category itself rather than the bicategory.

Garner and Gurshki [7] do something similar for tricategories, i.e. monoidal bicategories with several objects. Their locally cubical bicategories are like bicategories but where the homs are weak double categories rather than mere categories. The main feature, which makes the notion tractable, is the same: the coherence constraints are isomorphisms rather than equivalences. With a view to globularity the 2-cells in the strict direction (i.e. horizontal arrows in the hom double categories) are discarded, and with them the coherence isomorphisms. In the specific examples they consider these 2-cells have companions (as is almost always the case) and these are always equivalences. They show that, in this way, the resulting structure is in fact a tricategory.

In Section 3 we remark that monoidal double categories and locally cubical bicategories can be viewed as intercategories. One might say that rather than working with tricategories we should use intercategories instead or, more conservatively that there is a nice class of tricategories best studied using intercategory techniques.

In another direction, the coherence theorem for tricategories [8] says that they are triequivalent to Gray categories, and in Section 5 we show how Gray categories are special kinds of intercategories. One might envision a reworking of the coherence theorem by first showing that every tricategory is reducible to an intercategory in a nice way, and then proving a coherence theorem for intercategories, which should be relatively easy. This has yet to be carried out, but the point is that even if intercategories won't replace tricategories, they may prove to be a useful tool in their study.

The plan of the paper is as follows. After a short introductory section recalling the definition of intercategory, we consider duoidal categories as intercategories in Section 2. We also construct a non-trivial intercategory from a monoidal category with coproducts and show that certain duoidal categories can be embedded in as reflective subintercategories. The adjunction is of a sort proper to intercategories, and provides a nice example

of how different types of intercategories can be compared.

Section 3 is devoted to monoidal double categories and locally cubical bicategories as discussed above. We introduce two natural generalizations of monoidal double category, one called horizontal and the other vertical. A ready supply of these is given by the horizontal (resp. vertical) endomorphisms in an arbitrary intercategory.

Verity's double bicategories are considered in Section 4. His motivation was that in order to define double categories which are weak in both directions, examples of which came up in his thesis [21], it was necessary to give, as part of the structure, not only double cells but horizontal and vertical globular ones. The three kinds of cells suggest a triple category of some sort and indeed we show how to view double bicategories as intercategories with special properties. Among these special properties is that the horizontal and vertical cells have basic companions and conjoints, a property which deserves further attention. As double categories are in particular double bicategories, this shows how to consider a double category as an intercategory. Other useful ways of doing this are discussed here too.

Section 5 deals with Gray categories. As mentioned above, every tricategory is triequivalent to a Gray category. In trying to transform a tricategory into an equivalent one in which all the constraints were identities, [8] soon realized that it could not be done completely. They can all be made identities except for interchange which remains an isomorphism. The particular structure they got was called a Gray category. This is a category enriched in the monoidal category of 2-categories where the tensor encodes the interchange isomorphism, of a sort studied by Gray in [16]. But it was not exactly the tensor by Gray: where he had a comparison cell between certain composites they had an isomorphism. Now, failure of interchange is at the heart of intercategories (whence the name), so one might look to categories enriched in 2-categories with his original tensor as a source of intercategories. And indeed this works. Such enriched categories are shown to be special kinds of intercategory. But there is another tensor gotten by taking the afore mentioned comparison in the opposite direction. This was already noted in [8], but in their case the two were isomorphic, and they chose one arbitrarily. But for us the two are different and give different ways of considering (true) Gray categories as intercategories. But there is a much better way of viewing them as intercategories. There is enough room in an intercategory to accommodate both tensors. Thus we get a "symmetric" way of embedding Gray categories into intercategories where arbitrary choices are neither necessary nor possible.

Section 6 looks at variations on the theme of spans. Taking spans in a double category with the appropriate kind of pullback gives intercategories where interchange fails. If the double category is already spans, we get an intercategory whose basic cells are spans of spans which provide important examples. If we start with a double category of cospans we get an intercategory with spans of cospans as basic cells, something already occurring in the literature [5] though without the notion of intercategory it was impossible to formalize. We treat the notion of spans in the last section to get examples of intercategories where all the features are non-trivial.

The paper concludes with Section 7 where we study the intercategory whose basic cells are spans of spans of sets. We maintain that this should be considered to be the "intercategory of sets", Set. To back up this claim we study hom functors for an arbitrary intercategory A. These turn out to be lax-lax morphisms into Set. The proof of which uses all of the intercategory structure of A.

1. Preliminaries

In [13] we introduced intercategories and their morphisms and exposed their basic properties. We gave three equivalent presentations. The first as pseudocategories in the 2-category $\mathcal{L}x\mathcal{D}bl$ of weak double categories, lax functors and horizontal transformations. The second as pseudocategories in $\mathcal{C}x\mathcal{D}bl$, the 2-category of weak double categories, colax functors and horizontal transformations. In fact, it is better, as far as morphisms are concerned, to consider these as horizontal (and, respectively, vertical) pseudocategories in $\mathbb{D}bl$, the strict double category of weak double categories, lax functors, colax functors and their cells. These presentations are short and clear and an obvious generalization of duoidal categories, but a more intuitive presentation is as a double pseudocategory in $\mathcal{C}at$. A pseudocategory in $\mathcal{C}at$ is a weak double category, so this presentation shows an intercategory as two double categories sharing a common horizontal structure. This is like thinking of a double category as two categories with the same objects. Of course the two structures are related, which is where interchange appears.

At a more basic level, an intercategory A has a class of objects, and three kinds of arrows, horizontal, vertical and transversal each with their own composition (\circ, \bullet, \cdot) and identities (id, Id, 1, resp.). These are related in pairs by double cells as depicted in the diagram



Here the $h, h', \bar{h'}$ are horizontal, v, v', w' are vertical and f, \bar{f}, g transversal. Cells whose boundaries are horizontal and transversal, such as ϕ above, are called *horizontal*, those whose boundaries are vertical and transversal, such as ψ , are called *vertical*, and those like α' with horizontal and vertical boundaries are *basic*. Each of the three types of cells has two compositions like in a double category. In fact horizontal (resp. vertical) cells are the double cells of a weak double category. The fundamental unit of structure is the *cube*, as depicted above. Cubes have three compositions: horizontal, vertical and transversal. Transversal composition is strictly associative and unitary, giving four transversal categories. Horizontal and vertical composition are associative and unitary up to coherent transversal isomorphism.

The first feature of intercategories which distinguishes them from what one might imagine a weak triple category would be, is that both horizontal and vertical composition are bicategorical in nature, rather than having one of the composites associative and unitary up to equivalence as for tricategories. So in this sense they are a stricter notion. But in another sense they are laxer. The interchange law for basic cells doesn't hold. Instead there is a comparison, the interchanger

$$\chi: (\alpha \circ \beta) \bullet (\bar{\alpha} \circ \bar{\beta}) \longrightarrow (\alpha \bullet \bar{\alpha}) \circ (\beta \bullet \bar{\beta}).$$

 χ is a *special cube* meaning a cube whose horizontal and vertical faces are transversal identities. There will be many examples given below. The two-dimensional notation

$$\chi: \frac{\alpha|\beta}{\bar{\alpha}|\bar{\beta}} \longrightarrow \frac{\alpha}{\bar{\alpha}} \Big| \frac{\beta}{\bar{\beta}}$$

is often used. In it the variables (cells) don't change place. There are also *degenerate interchangers*:

$$\mu : \frac{\mathrm{id}_{v}}{\mathrm{id}_{\bar{v}}} \longrightarrow \mathrm{id}_{\frac{v}{\bar{v}}}$$
$$\delta : \mathrm{Id}_{h|h'} \longrightarrow \mathrm{Id}_{h}|\mathrm{Id}_{h'}$$
$$\tau : \mathrm{Id}_{\mathrm{id}_{A}} \longrightarrow \mathrm{id}_{\mathrm{Id}_{A}}$$

All these satisfy a number of coherence conditions, which can be found in [13].

If all interchangers are identities as well as the associativity and unit isomorphisms, we have a triple category. If they are all isomorphisms we talk of *weak triple category*. A case of special importance is when the δ , μ , τ are identities while the χ is allowed to be arbitrary. We call this a chiral triple category. It will play a central role in [15].

There are three general types of morphism of intercategory. They all preserve the transversal structure on the nose. In the horizontal and vertical directions they can be lax or colax. We can have laxity in both directions, which we call lax-lax morphisms. Similarly there are colax-colax morphisms. The colax-lax morphisms are colax in the horizontal direction and lax in the vertical. The lax-colax doesn't come up. In fact the obvious coherence conditions produce diagrams in which none of the arrows compose.

2. Duoidal categories

2.1. Duoidal categories as intercategories.

Duoidal categories were introduced in [1] under the name of 2-monoidal categories as a generalization of braided monoidal categories and motivated by various kinds of morphisms between these.

The classical Eckmann-Hilton argument says that a monoid in the category of monoids is a commutative monoid and we might think then that a pseudomonoid in the 2-category of monoidal categories and strong monoidal functors could be, for similar reasons, a symmetric monoidal category. This is not quite true. What emerges is the important notion of braided monoidal category as exposed in the now classical paper [17].

If instead we consider pseudomonoids in the 2-category of monoidal categories and lax monoidal functors we get categories equipped with two tensor products related by interchange morphisms. These morphisms express the fact that the second tensor is given by a lax functor with respect to the first, but could equally well be understood as saying that the first tensor is colax with respect to the second in a way that reminds us of the definition of bialgebra. This is the notion of *duoidal category* (or 2-monoidal category).

Duoidal categories have been studied (apart from *loc. cit.*) in [4, 3], where many examples are given.

Our notion of intercategory is partly modeled after this, so it will be no surprise that duoidal categories can be considered as special intercategories just as monoidal categories can be viewed as one-object bicategories. However, it is perhaps not in the first way one might think.

Definition 2.2 of [13] says that an intercategory is a pseudocategory in $\mathcal{L}x\mathcal{D}bl$, the 2-category of weak double categories with lax functors and horizontal transformations. A monoidal category may be considered as a weak double category with one object and one horizontal morphism, the identity. Then lax functors are lax monoidal functors and horizontal transformations are monoidal natural transformations. So we have a full sub 2-category $\mathcal{L}x\mathcal{M}on$ of $\mathcal{L}x\mathcal{D}bl$. Also, a pseudomonoid is a pseudocategory whose object of objects is 1. In this way a duoidal category \mathbf{D} , which is a pseudomonoid in $\mathcal{L}x\mathcal{M}on$, can be considered as a special intercategory. It will have one object, only identity horizontal, vertical and transversal arrows, and the horizontal and vertical cells are also identities. The only nontrivial parts are the basic cells which are the objects of \mathbf{D} and the cubes which are its morphisms. A general cube will look like



with a morphism of \mathbf{D} , $d: D \longrightarrow D'$, in it. The first tensor gives horizontal composition and the second tensor, the vertical.

As a double pseudocategory in CAT, as described in Section 3 of [13], it is



Furthermore, the bilax, double lax and double colax morphisms of [1] correspond to our colax-lax, lax-lax and colax-colax functors.

A ready supply of duoidal categories can be gotten from monoidal categories (\mathbf{V}, \otimes, I) with finite products. Indeed, $(\mathbf{V}, \times, 1, \otimes, I)$ is Example 6.19 of [1]. (Note however that, contrary to *loc. cit.*, we list the horizontal structure, product here, first.) No coherence between \otimes and product is assumed. Dually if \mathbf{V} has finite coproducts, then $(\mathbf{V}, \otimes, I, +, 0)$ is a duoidal category. In particular, for any category \mathbf{A} with finite products and coproducts, we get a duoidal category $(\mathbf{A}, \times, 1, +, 0)$.

2.2. MATRICES IN A MONOIDAL CATEGORY.

A closely related intercategory is the following. Let \mathbf{V} be a monoidal category with coproducts preserved by \otimes in each variable, and with pullbacks. We construct an intercategory $\mathsf{SM}(\mathbf{V})$ whose objects are sets, whose transversal arrows are functions, whose horizontal arrows are spans, and whose vertical arrows are matrices of \mathbf{V} objects. Specifically, a vertical arrow $A \longrightarrow B$ is an $A \times B$ matrix $[V_{ab}]$ of objects V_{ab} of \mathbf{V} . Horizontal cells are span morphisms and vertical cells are matrices of morphisms of \mathbf{V} . A basic cell is a span of matrices

where

$$V_{\sigma_0 s, \tau_0 t} \xleftarrow{f_{st}} W_{st} \xrightarrow{g_{st}} V'_{\sigma_1 s, \tau_1 t}$$

are morphisms of \mathbf{V} . A general cube



is a morphism of spans of matrices, i.e.



forming two commutative cubical diagrams.

In Section 6 we shall give a general construction showing, in particular, that this is indeed an intercategory. Unless \otimes preserves pullback, the interchanger χ is not invertible. The identities are as follows. The horizontal identity $\mathrm{id}_{[V_{ab}]}$ is

These compose vertically so

$$\mu: \frac{\mathrm{id}_{[V_{ab}]}}{\mathrm{id}_{[W_{bc}]}} \longrightarrow \mathrm{id}_{[V_{ab}]\otimes[W_{bc}]}$$

is equality. The vertical identity Id_S is

ne vertilear identity ids is



where $\mathrm{Id}_X : X \longrightarrow X$ is given by

$$(\mathrm{Id}_X)_{x,x'} = \begin{cases} I & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases}$$

and for $f: X \longrightarrow Y$



is given by

$$(\mathrm{Id}_f)_{x,x'} = \begin{cases} 1_I : I \longrightarrow I & \text{if } x = x' \\ ! : & 0 \longrightarrow I & \text{if } x \neq x' & \text{and } fx = fx' \\ 1_0 : 0 \longrightarrow 0 & \text{if } fx \neq fx'. \end{cases}$$

The horizontal composition $Id_S|Id_{S'}$ will usually involve the pullback



and unless this is 0 (i.e. $0 \longrightarrow I$ is mono), $\delta : \mathrm{Id}_{S \otimes S'} \longrightarrow \mathrm{Id}_S | \mathrm{Id}_{S'}$ will not be invertible. Finally $\tau : \mathrm{Id}_{\mathrm{Id}_A} \longrightarrow \mathrm{Id}_{\mathrm{Id}_A}$ is always the identity.

By contrast, all of the interchangers χ, μ, δ, τ are generally not invertible for the cartesian product/tensor duoidal category of Subsection 2.1.

2.3. Embedding the duoidal category of V into matrices.

If **V** has a terminal object, we can embed $(\mathbf{V}, \times, \otimes)$, considered as an intercategory, into $\mathsf{SM}(\mathbf{V})$ as follows. A basic cell of **V** is embedded as

The extension in the transversal direction is obvious. V cannot be a strict subintercategory of $\mathsf{SM}(\mathbf{V})$ as this would imply that μ and τ for V are identities. Indeed, the vertical arrow [1] : 1 \longrightarrow 1 is the 1 × 1 matrix whose sole entry is 1, the terminal object of V, whereas the vertical identity is the one whose entry is *I*, the unit for \otimes . What we have is an inclusion $F : \mathbf{V} \longrightarrow \mathsf{SM}(\mathbf{V})$ which is strong in the horizontal direction and lax in the vertical direction. So it can be considered as a lax-lax or a colax-lax morphism.

There is also a morphism in the opposite direction, $G: \mathsf{SM}(\mathbf{V}) \longrightarrow \mathbf{V}$, taking a basic cell

to the cell

with the obvious extension in the transversal direction. G is colax-colax. For example, the identity structure morphisms are

$$G(\mathrm{id}_{[U_{ab}]}) = \sum_{a,b} U_{a,b} \longrightarrow 1$$

and

$$G(\mathrm{Id}_S) = \nabla : \sum_S I \longrightarrow I$$

the codiagonal. G is left adjoint to F in the following sense. F may be considered as a lax-lax morphism or a colax-lax morphism, i.e. as a horizontal or a transversal arrow in **ICat**, the triple category of intercategories. As transversal arrows are generally better let's consider F as such. Then F and G are horizontal and vertical arrows in the double category of transversal and vertical arrows of **ICat**, i.e. in $\mathbb{PsCat}(\mathcal{CxDbl})$; moreover, F and G are conjoint arrows in the latter.

To see this we need double cells



satisfying the "triangle equalities". Such double cells take objects, horizontal and vertical arrows, and basic cells to transversal arrows, horizontal and vertical cells, and cubes respectively. GF is the identity on all elements and $\alpha : GF \longrightarrow id \cdot Id$ is taken to be the appropriate identity.

The various components of $\beta: \mathrm{Id} \cdot \mathrm{id} \longrightarrow F \cdot G$ can be read off from its action on a basic cell



which produces the cube



where the middle cell is given by the coproduct injections



(All the other morphisms are uniquely determined.)

Checking that α and β are double cells and that they satisfy the conjoint equations is straightforward and omitted.

3. Monoidal double categories and cubical bicategories

3.1. MONOIDAL DOUBLE CATEGORIES.

In [20], Shulman uses a notion of monoidal double category to construct monoidal bicategories. The notion of monoidal double category is simpler because the coherence morphisms are isomorphisms rather than equivalences, which makes the coherence conditions much easier. In *loc. cit.* many examples are given building a strong case for the point of view that the seemingly more complicated notion of double category is in fact simpler than that of bicategory.

A monoidal double category [20] is a pseudomonoid in the 2-category StgDbl of (weak) double categories with strong functors and horizontal transformations:

$$\otimes: \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D},$$
$$I: \mathbb{1} \longrightarrow \mathbb{D}.$$

As $Stg\mathcal{D}bl$ is a sub 2-category of $\mathcal{L}x\mathcal{D}bl$ (and $\mathcal{C}x\mathcal{D}bl$) and intercategories involve only pullbacks of strict double functors, which are in $Stg\mathcal{D}bl$, it follows that a weak category object in $Stg\mathcal{D}bl$ is also one in $\mathcal{L}x\mathcal{D}bl$ (and $\mathcal{C}x\mathcal{D}bl$), i.e. an intercategory. It is one in which the interchangers χ , μ , δ , τ are all isomorphisms. So a monoidal double category is an intercategory of the form



with strong interchangers (isomorphisms).

It is an intercategory with one object, one transversal arrow, one vertical arrow and one vertical cell, all identities of course.

Furthermore interchange holds up to isomorphism. As a double pseudocategory in CAT, it looks like



The conditions (iv) of *loc. cit.* correspond to (24), (26), (25) in [13, Sect. 4], conditions (v) to (27), (28), and conditions (vi) to (31), (30), (29), (32). Our conditions (21), (22), (23) don't appear because the structural isomorphisms of the double category \mathbb{D} were treated as identities.

A monoidal double category can equally well be viewed as an intercategory with one object, one transversal arrow, one horizontal arrow and one horizontal cell by using the inverse interchangers. The 3×3 diagram of categories would then be the transpose of the above.

3.2. HORIZONTAL AND VERTICAL MONOIDAL DOUBLE CATEGORIES.

In the present context, it seems natural to remove the restriction that the interchangers be isomorphisms. We then get two separate notions of monoidal double category corresponding to the cases just mentioned. One in which $\otimes : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}$ and $I : \mathbb{1} \longrightarrow \mathbb{D}$ are lax, which we call *horizontal monoidal double category*, and the other where \otimes and Iare colax, which we call *vertical*. Let us examine this in more detail. For notational convenience we look at vertical monoidal double categories. That \otimes and I are colax means that we have comparison cells



satisfying the conditions (21)-(32) of Section 3 in [13].

This definition encompasses duoidal categories. Another example is a double category with a lax choice of finite products, as in [10].

3.3. ENDOMORPHISMS IN AN INTERCATEGORY.

Just like the set of endomorphisms of an object in a category has a monoid structure, if we fix an object A of an intercategory A we get two monoidal double categories of endomorphisms, a horizontal one $\mathbb{HE}nd(A)$ and a vertical one $\mathbb{VE}nd(A)$ (or $\mathbb{HE}nd_A(A)$ and $\mathbb{VE}nd_A(A)$ if there are several intercategories). As an intercategory, $\mathsf{HE}nd(A)$ (or $\mathsf{HE}nd_A(A)$) has the same structure as A except that we only consider the one object Aas well as only the identity transversal arrow 1_A , the identity vertical arrow Id_A , and the identity vertical cell 1_{Id_A} . So a general cube would be an A cube that looks like



As a monoidal double category, $\mathbb{HE}nd(A)$ has objects the horizontal endomorphisms of A, horizontal arrows the horizontal cells, and vertical arrows the basic cells. The tensor product is given by horizontal composition. This indeed gives us what we are calling a horizontal monoidal double category. It will only be a monoidal double category in the sense of [20] if the interchangers χ, δ, μ, τ are isomorphisms when restricted to basic cells of the form



The construction of $\mathbb{VEnd}(A)$ is dual, and considers only cubes of the form



This produces a vertical monoidal double category.

3.4. MATRICES IN A DUOIDAL CATEGORY.

We outline an interesting example of a horizontal monoidal double category constructed from a duoidal category $(\mathbf{D}, \otimes, I, \boxtimes, J)$ having coproducts over which \boxtimes distributes. The double category \mathbb{D} has sets as objects, functions as horizontal arrows, matrices of **D**-objects as vertical arrows and matrices of **D**-morphisms as cells. Vertical composition is given by matrix multiplication using \boxtimes , and vertical identities Id are "scalar matrices" with J on the diagonal. This is what we called **V**-Set in [19] with $\mathbf{V} = (\mathbf{D}, \boxtimes, J)$.

The tensor product $\otimes : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}$ is cartesian product on objects (sets) and horizontal arrows (functions). For vertical arrows, it is defined pointwise using the \otimes of **D**



with the obvious extension to cells. The unit for \otimes is the 1 \times 1 matrix with entry *I*.

The laxity morphisms of \otimes are as follows. Suppose $[W_{bc}] : B \longrightarrow C$ and $[W'_{b'c'}] : B' \longrightarrow C'$ are two more vertical arrows of \mathbb{D} . Then

$$\chi: (V \otimes V') \cdot (W \otimes W') \longrightarrow (V \cdot W) \otimes (V' \otimes W')$$

has as its (a, a'), (c, c') component the composite

$$\sum_{(b,b')} (V_{ab} \otimes V'_{a'b'}) \boxtimes (W_{bc} \otimes W'_{b'c'}) \xrightarrow{\sum_{(b,b')} \chi}$$
$$\sum_{(b,b')} (V_{ab} \boxtimes W_{bc}) \otimes (V'_{a'b'} \boxtimes W'_{b'c'}) \xrightarrow{[j_b \otimes j_{b'}]}$$
$$(\sum_{b} V_{ab} \boxtimes W_{bc}) \otimes (\sum_{b'} V'_{a'b'} \boxtimes W'_{b'c'})$$

where j_b , $j_{b'}$ are coproduct injections.

The (a, b), (a', b') component of

$$\delta: \mathrm{Id}_{A \times B} \longrightarrow \mathrm{Id}_A \otimes \mathrm{Id}_B$$

is given by

$$\begin{split} \delta: J &\longrightarrow J \otimes J \quad if \quad a = a', b = b' \\ !: 0 &\longrightarrow J \otimes 0 \quad if \quad a = a', b \neq b' \\ !: 0 &\longrightarrow 0 \otimes J \quad if \quad a \neq a', b = b' \\ !: 0 &\longrightarrow 0 \otimes 0 \quad if \quad a \neq a', b \neq b' \end{split}$$

The laxity morphisms for $I : \mathbb{1} \longrightarrow \mathbb{D}$ are given by

$$\mu: I \boxtimes I \longrightarrow I$$

and

$$\tau: J \longrightarrow I.$$

The routine calculations showing that we actually get a horizontal monoidal double category are omitted.

3.5. LOCALLY CUBICAL BICATEGORIES.

A multiobject version of monoidal double categories is given by Garner and Gurski's locally cubical bicategories [7]. These are categories weakly enriched in the monoidal (cartesian) 2-category $Stg\mathcal{D}bl$. So a class Ob \mathcal{A} of objects is given, and for each pair $A, B \in Ob\mathcal{A}$ a weak double category $\mathcal{A}(A, B)$. For each object A there is given a strong functor

$$\mathrm{Id}_A:\mathbb{1}\longrightarrow\mathcal{A}(A,A)$$

and for any three objects A, B, C, a strong functor

• :
$$\mathcal{A}(A, B) \times \mathcal{A}(B, C) \longrightarrow \mathcal{A}(A, C).$$

This composition is unitary and associative up to coherent isomorphism (see *loc. cit.* for details).

One can get a good feel for this structure by considering the *category* **StctDbl** of strict double categories and strict functors. These are category objects and their functors in **Cat** and so form a cartesian closed category. That is, for any two double categories \mathbb{A} and \mathbb{B} we have a double category $\mathbb{B}^{\mathbb{A}}$ of morphisms from \mathbb{A} to \mathbb{B} . One can easily work out what $\mathbb{B}^{\mathbb{A}}$ looks like. Its objects are strict functors, its horizontal arrows are the horizontal transformations we have been using, its vertical arrows are the dual notion of vertical transformation, and its cells are double transformations. This makes **StctDbl** into a category enriched in itself, i. e. a strict locally cubical bicategory.

Returning to the non strict case, we can combine the whole structure into a pseudocategory in StgDbl:

$$\sum_{A,B,C} \mathcal{A}(A,B) \times \mathcal{A}(B,C) \xrightarrow{p_1} \sum_{A,B} \mathcal{A}(A,B) \xrightarrow{\partial_0} \operatorname{Ob} \mathcal{A}$$

where $Ob\mathcal{A}$ is a discrete double category.

Thus we see that a locally cubical bicategory is an intercategory in which the only transversal and vertical arrows are identities as well as vertical cells, and for which the interchangers are isomorphisms. A general cube looks like



with a double cell of $\mathcal{A}(A, B)$ inside it.

4. Verity double bicategories

4.1. DOUBLE BICATEGORIES.

Double bicategories are, at least in part, an answer to the problem of making double categories weak in both directions. For example, we could take quintets in a bicategory \mathcal{B} . This structure has the same objects as \mathcal{B} with horizontal and vertical arrows the arrows of \mathcal{B} and with double cells diagrams



where $t: kf \longrightarrow gh$ is a 2-cell. Such cells can be pasted horizontally and vertically, and everything works well (including interchange) except that neither horizontal nor vertical composition is associative or unitary on the nose.

A simpler example is the transpose of a weak double category, where horizontal and vertical are interchanged. This is a useful duality for strict double categories but is not available for weak ones.

Attempts at a direct definition of double categories, weak in both directions, just lead to vicious circles. The problem lies with the special cells used in the coherence conditions for the definition of weak double category. These are cells whose vertical domains and codomains are horizontal identities, but if these identities are not strict identities, then horizontal composition of special cells would require the use of vertical special cells, and now the same problem arises. The resolution is achieved by formalizing special cells. This is done by giving as extra structure, cells between arrows whose domains (and codomains) are the *same*, i.e. *globular cells* as well as the double ones. Although the special cells involved in the definition of weak double category are all isomorphisms, non invertible ones come up in the definition of lax and colax functor.

We sketch Verity's definition of double bicategory. The reader is referred to [21] for details. Section 3.2 of [18] also gives a very readable account.

To start with we are given two bicategories \mathcal{H} and \mathcal{V} which share the same class of objects A and then we are given a class of squares \mathcal{S} with boundaries



 h, \bar{h} arrows of \mathcal{H} and v, v' arrows of \mathcal{V} . There are furthermore left and right actions of the 2-cells of \mathcal{V} on the σ and top and bottom actions of those of \mathcal{H} on them as well, e.g.



These four actions commute (strictly). Finally the squares can be pasted horizontally and vertically. Horizontal and vertical pasting is associative and unitary once the structural isomorphisms of \mathcal{H} (or \mathcal{V}) are factored in so as to make domains and codomains agree. The interchange law for squares holds strictly.

We already have the beginning of an intercategory



H (resp. **V**) is the category of arrows and 2-cells of \mathcal{H} (resp. \mathcal{V}). **S** is the category whose objects are squares with morphisms described below.

Given a double bicategory $(A, \mathcal{H}, \mathcal{V}, \mathcal{S}, \ldots)$ we construct an intercategory D as follows.

- (1) The objects are the elements of A.
- (2) The transversal arrows are identities.
- (3) The horizontal (vertical) arrows are the arrows of \mathcal{H} (resp. \mathcal{V}).
- (4) The horizontal (vertical) cells are the 2-cells of \mathcal{H} (resp. \mathcal{V}).
- (5) The basic cells are the elements of \mathcal{S} .

(6) There is a single cube with boundary as below



otherwise there are none.

This last condition tells us what the morphisms of **S** are: a morphism $\sigma \longrightarrow \sigma'$ is a quadruple $(\alpha, \beta, \bar{\alpha}, \beta')$ as above.

The interchangers χ , δ , μ , τ are all identities.

Apart from the fact that transversal arrows are all identities, there is a more important special feature of intercategories D arising in this way:



is a discrete bifibration. Let's call this property (*). It implies, in particular, that every horizontal and every vertical cell has a basic companion and conjoint. It also implies that the interchangers are identities.

4.1.1. THEOREM. There is a natural correspondence between double bicategories and intercategories satisfying (*) and whose transversal arrows are identities.

4.2. Double categories as intercategories.

One thing this example gives us is a different way of looking at intercategories. They are a weakening of double categories so as to allow both horizontal and vertical composition to be bicategorical in nature. And thus it gives us a preferred way to consider a double category as an intercategory.

Let A be a weak double category. Horizontal composition is strictly associative and unitary whereas vertical composition is so only up to coherent isomorphism. This is reflected in the fact that morphisms of double categories can be lax, colax, strong or strict in the vertical direction but are always required to be strict in the horizontal direction. To encode this in a Verity double bicategory we take \mathcal{H} to be the locally discrete bicategory (i.e. just the category) of objects and horizontal arrows of A. For \mathcal{V} we take the bicategory of objects, vertical arrows and special (= globular) cells of A. The class of squares \mathcal{S} is

if

the class of all double cells of \mathbb{A} . We now turn this into an intercategory $I(\mathbb{A})$ thus placing it in the same environment (i. e. the triple category ICat) as the other examples. Thus we have for $I(\mathbb{A})$

- objects are those of \mathbb{A}
- transversal arrows are identities
- horizontal arrows are those of \mathbb{A}
- vertical arrows are those of \mathbb{A}
- horizontal cells are identities
- vertical cells are special cells of \mathbb{A}
- basic cells are the double cells of \mathbbm{A}
- cubes are commutative cylinders $\phi' \alpha = \beta \phi$



The three kinds of morphisms of intercategory $I(\mathbb{A}) \longrightarrow I(\mathbb{B})$, lax-lax, colax-lax, colax-colax, correspond respectively to lax, lax, colax functors $\mathbb{A} \longrightarrow \mathbb{B}$.

Of course there are other ways of considering a double category as an intercategory. The two examples mentioned at the beginning of the section, quintets in a bicategory and the transpose of a weak double category, require horizontal special cells as well. We would take the \mathcal{H} to be the bicategory of objects, horizontal and special cells of \mathbb{A} , with \mathcal{V} and \mathcal{S} the same as above. The new intercategory $I'(\mathbb{A})$ will have cubes that involve six cells



making the cube commute, i.e.

We will not check the tedious though straightforward details showing that this is indeed an intercategory.

The above example suggests the following generalization which does not, however, arise as a Verity double bicategory. From a weak double category \mathbb{A} we construct an intercategory $\mathsf{I}''(\mathbb{A})$ which is like $\mathsf{I}'(\mathbb{A})$ except that we allow its transversal morphisms to be horizontal arrows of \mathbb{A} . So a general cube will look like



such that



4.3. QUINTETS IN A DOUBLE CATEGORY.

We end this section with a somewhat dual construction to the previous one, quintets in a double category. A weak double category may be thought of as a bicategory (vertically) with some extra arrows (horizontal) which serve to rigidify it in a sense. The quintet construction for bicategories mentioned above can be performed on an arbitrary weak double category. Let $\mathbb A$ be a weak double category. The intercategory of quintets, $\mathsf Q(\mathbb A),$ has the following:

- objects, those of \mathbb{A}
- transversal arrows, the horizontal arrows of \mathbbm{A}
- horizontal and vertical arrows, the vertical arrows of \mathbbm{A}
- horizontal and vertical cells, the double cells of \mathbbm{A}

– basic cells



are quintets, i.e. special cells of \mathbbm{A}



– cubes consist of cells as follows



such that



Again we omit the straightforward verifications.

5. True Gray categories

5.1. GRAY'S ORIGINAL TENSOR.

Gray categories came into prominence with the work of Gordon, Power and Street [8] on tricategories. Whereas every bicategory is biequivalent to a 2-category, the corresponding result for tricategories, that they be triequivalent to 3-categories, is false as this would imply, as a special case, that every symmetric monoidal category is equivalent to a strict one. Their coherence result was that every tricategory is triequivalent to one in which everything is strict except for interchange which only held up to isomorphism, a notion they called "Gray category". In fact, they introduced a monoidal structure on the category of 2-categories which encodes this failure of interchange. The resulting monoidal category they called **Gray**, and a Gray category is a category enriched in **Gray**.

As the name suggests, this was strongly influenced by a similar monoidal structure introduced by Gray in [16]. His tensor product encodes the possibility that interchange "hold" only up to a comparison morphism. It arose, via adjointness, from a natural internal hom on the category of 2-categories, which we briefly outline.

We consider the category 2-**Cat** of 2-categories and 2-functors. Between 2-functors we have various kinds of transformations, of which lax (natural) transformations are an important class. A lax transformation $t: F \longrightarrow G$, for $F, G: \mathcal{A} \longrightarrow \mathcal{B}$ 2-functors, is given by

(1) $tA: FA \longrightarrow GA$ for each object A

(2) a 2-cell



for each arrow $f: A \longrightarrow A'$ in \mathcal{A} . These satisfy well known conditions [2]. Between lax

transformations, there are modifications $\mu: t \longrightarrow u$ given by 2-cells

$$FA$$
 $\downarrow \mu A$ GA uA

again satisfying obvious conditions. In this way we get a 2-category $\operatorname{Fun}(\mathcal{A}, \mathcal{B})$, an internal hom for 2-**Cat**.

But it doesn't make 2-Cat cartesian closed because composition

$$\operatorname{Fun}(\mathcal{A},\mathcal{B}) \times \operatorname{Fun}(\mathcal{B},\mathcal{C}) \longrightarrow \operatorname{Fun}(\mathcal{A},\mathcal{C})$$

isn't a 2-functor. Composition of 2-functors poses no problem. But for lax transformations



we have two possible choices for $(vt)A : HFA \longrightarrow KGA$, either the top or bottom composite in

$$\begin{array}{c|c} HFA \xrightarrow{vFA} KFA \\ \downarrow \\ HtA \\ \downarrow \\ HGA \xrightarrow{vGA} KGA \end{array}$$

Each choice extends to a lax transformation via

$$HFA \xrightarrow{vFA} KFA \xrightarrow{KtA} KGA$$

$$\downarrow Ff \qquad \qquad \downarrow vFf \qquad \qquad \downarrow KFf \qquad \qquad \downarrow Ktf \qquad \qquad \downarrow KGf$$

$$HFA' \xrightarrow{vFA'} KFA' \xrightarrow{KtA'} KGA'$$

and

$$HFA \xrightarrow{HtA} HGA \xrightarrow{vGA} KGA$$

$$HFf \qquad \qquad \downarrow Htf \qquad \qquad \downarrow HGf \qquad \qquad \downarrow vGf \qquad \qquad \downarrow KGf$$

$$HFA' \xrightarrow{HtA'} HGA' \xrightarrow{vGA'} KGA'$$

and each of these composites is associative and unitary, and functorial with respect to modifications. But neither satisfies interchange. Whiskering, on the other hand, works well as there is no interchange involved, and the two composites come from that in the standard way. There is furthermore a comparison between the two. Clearly there is a lot of nice structure here and it is a question of organizing it properly. The key to this is Gray's tensor product, obtained from Fun by adjointness.

We would like a 2-category $\mathcal{A} \otimes \mathcal{B}$ so that there is a 2-natural bijection

$$\frac{2\text{-functors} \quad \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}}{2\text{-functors} \quad \mathcal{B} \longrightarrow \operatorname{Fun}(\mathcal{A}, \mathcal{C})}$$

Analyzing what a 2-functor $\mathcal{B} \longrightarrow \operatorname{Fun}(\mathcal{A}, \mathcal{C})$ is, we get what we shall call a *Gray functor* of two variables $H : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$, i.e.

- (1) a 2-functor $H(A, -) : \mathcal{B} \longrightarrow \mathcal{C}$ for every A in \mathcal{A} ;
- (2) a 2-functor $H(-,B): \mathcal{A} \longrightarrow \mathcal{C}$ for every B in \mathcal{B} ;
- (3) H(A, -)(B) = H(-, B)(A), written H(A, B);
- (4) for every $f: A \longrightarrow A'$ and $g: B \longrightarrow B'$ a 2-cell

$$\begin{array}{c|c} H(A,B) \xrightarrow{H(f,B)} H(A',B) \\ & \xrightarrow{H(A,g)} & \xrightarrow{h(f,g)} & \downarrow \\ H(A,g) & \xrightarrow{H(f,g)} H(A',g) \\ & \xrightarrow{H(A,B')} H(A',B') \end{array}$$

satisfying compatibility conditions for composition of the f's (and the g's).

Gray called these *quasi-functors of two variables*, which clashes with the now accepted use of quasi. They are also sometimes called *cubical functors*, which presents the same problem.

It is easy to imagine what $\mathcal{A} \otimes \mathcal{B}$ is. It is the free 2-category with pairs (A, B), Ain \mathcal{A}, B in \mathcal{B} , as objects, arrows generated by $(f, B) : (A, B) \longrightarrow (A', B)$ and (A, g) : $(A, B) \longrightarrow (A, B')$ subject to the equations

$$(f', B)(f, B) = (f'f, B)$$

 $(A, g')(A, g) = (A, g'g)$
 $(1_A, B) = 1_{(A,B)} = (A, 1_B).$

The 2-cells are generated by those of \mathcal{A} , those of \mathcal{B} , and formal cells

$$(A, B) \xrightarrow{(f,B)} (A', B)$$

$$(A,g) \downarrow \xrightarrow{\gamma(f,g)} \qquad \downarrow (A',g)$$

$$(A, B') \xrightarrow{(f,B')} (A', B')$$

subject to the expected equations. See [16] for a more detailed description. It is of course complicated but just knowing it exists and that it gives a monoidal structure on 2-Cat

is enough. It is easier to use its universal property as classifying Gray functors of two variables. This monoidal structure is biclosed, with $\operatorname{Fun}(\mathcal{A}, -)$ being the right adjoint to $\mathcal{A} \otimes (-)$. The other adjoint is $\operatorname{Fun}^*(\mathcal{B}, -)$ given by

$$\operatorname{Fun}^*(\mathcal{B},\mathcal{C}) = \operatorname{Fun}(\mathcal{B}^{co},\mathcal{C}^{co})^{co}.$$

Fun^{*}(\mathcal{B}, \mathcal{C}) has 2-functors as objects, colax transformations as arrows, and modifications as 2-cells.

5.1.1. DEFINITION. We call a category enriched in 2-Cat with this tensor a *true Gray* category.

Thus a true Gray category has objects, arrows (1-cells), 2-cells and 3-cells with domains and codomains like for 3-categories. There is a strictly associative and unitary composition of arrows. 2-cells and 3-cells compose well inside the hom 2-categories, but there is no horizontal composition of 2-cells. Only whiskering on both sides by arrows, related by 3-cells as above. This last aspect suits our purposes well as a measure of the failure of interchange. But we do need composition of 2-cells and 3-cells across the hom 2-categories.

As hinted at above, there are two related ways of getting a composition, a lax and a colax one. The roots of this lie in the following result, which is essentially Gray's I.4.8 [16], the idea for which he credits Mac Lane.

5.1.2. PROPOSITION. There is a canonical bijection between the following three notions: (a) Gray functors of two variables $H : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$,

(b) lax functors $H^{\wedge} : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ for which the laxity morphisms

 $(i) \ H^{\wedge}(f,1)H^{\wedge}(f',g') \longrightarrow H^{\wedge}(ff',g')$

 $(ii) \ H^{\wedge}(f,g)H(1,g') \longrightarrow H^{\wedge}(f,gg')$ $(iii) \ 1_{H^{\wedge}(A,B)} \longrightarrow H^{\wedge}(1_A,1_B)$

are identities,

(c) colax functors $H^{\vee}: \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ for which the colaxity morphisms

 $\begin{array}{l} (i) \ H^{\vee}(f',gg') \longrightarrow H^{\vee}(1,g)H^{\vee}(f',g') \\ (ii) \ H^{\vee}(ff',g) \longrightarrow H^{\vee}(f,g)H^{\vee}(f',1) \\ (iii) \ H^{\vee}(1_A,1_B) \longrightarrow 1_{H^{\vee}(A,B)} \\ are \ identities. \end{array}$

Furthermore, Gray transformations (a.k.a. quasi-natural transformations) $H \longrightarrow K$ are in bijection with lax transformations $H^{\wedge} \longrightarrow K^{\wedge}$ and also with lax transformations $H^{\vee} \longrightarrow K^{\vee}$. A similar statement applies to modifications.

PROOF. (Sketch) $H^{\wedge}(A, B) = H^{\vee}(A, B) = H(A, B),$ $H^{\wedge}(f, g) = H(f, B)H(A', g),$ $h^{\wedge}(f, g; f', g') = H(f, B)h(f', g)H(A'', g),$ $H^{\vee}(f, g) = H(A, g)H(f, B'),$ $h^{\vee}(f, g; f, g) = H(A, g)h(f, g')H(f', B''),$ $H(A, g) = H^{\wedge}(1_A, g) = H^{\vee}(1_A, g),$ 5.2. Gray categories as intercategories – lax case.

It follows from the proposition that composition in a true Gray category may be considered as a special kind of lax functor

$$\mathcal{A}(A,B) \times \mathcal{A}(B,C) \longrightarrow \mathcal{A}(A,C)$$

or, alternatively, as a colax functor. In this way we define the horizontal composition of 2-cells and 3-cells in two different ways. Thus we get two different types of "3-category" with lax or colax interchange. In [8], two different tricategories are gotten from a **Gray** category, which are called left and right, but are equivalent and one is chosen arbitrarily. For true Gray categories the two are quite different, the one represented vertically, the other horizontally as intercategories.

Consider first the lax case. We take 2-categories as vertical double categories, which is forced because that is where the laxity occurs. Then we put all the homs together and get a category object

$$\sum_{A,B,C\in\operatorname{Ob}(\mathcal{A})} \mathcal{A}(A,B) \times \mathcal{A}(B,C) \xrightarrow[p_2]{p_1} \sum_{A,B\in\operatorname{Ob}(\mathcal{A})} \mathcal{A}(A,B) \xrightarrow[\partial_1]{e_1} \operatorname{Ob}(\mathcal{A})$$

in $\mathcal{L}x\mathcal{D}bl$, which is of course an intercategory.

Referring to the table of Section 4 of [13], we see that a true Gray category \mathcal{A} gives an intercategory A_l as follows:

- (1) The objects are those of \mathcal{A}
- (2) Transversal arrows are identities
- (3) Horizontal arrows are the 1-cells of \mathcal{A}
- (4) Vertical arrows are identities
- (5) Horizontal cells are identities
- (6) Vertical cells are identities
- (7) Basic cells are the 2-cells of \mathcal{A}
- (8) Cubes are the 3-cells of \mathcal{A} .

So a general cube would look like



with an α in the back face and a 3-cell $\alpha \longrightarrow \alpha'$ inside, corresponding to



in \mathcal{A} .

Transversal and vertical composition come from the 2-category homs of \mathcal{A} . Horizontal composition of arrows is that of \mathcal{A} , but for basic cells and cubes the decision was made, when we chose the lax case, to take

extended to cubes in the obvious way.

For interchange on

$$A \xrightarrow{f} B \xrightarrow{h} C$$

$$\| \downarrow \alpha \| \downarrow \beta \|$$

$$A \xrightarrow{g} B \xrightarrow{k} C$$

$$\| \downarrow \bar{\alpha} \| \downarrow \bar{\beta} \|$$

$$A \xrightarrow{g} B \xrightarrow{k} C$$

$$\| \downarrow \bar{\alpha} \| \downarrow \bar{\beta} \|$$

$$A \xrightarrow{l} B \xrightarrow{m} C$$

$$(\alpha \circ \beta) \bullet (\bar{\alpha} \circ \bar{\beta}) = \alpha h \cdot g\beta \cdot \bar{\alpha} k \cdot l\bar{\beta}$$

and

$$(\alpha \bullet \bar{\alpha}) \circ (\beta \bullet \bar{\beta}) = \alpha h \cdot \bar{\alpha} h \cdot l\beta \cdot l\bar{\beta}$$

Grayness of composition gives a 2-cell in $\mathcal{A}(A, C)$

$$x: g\beta \cdot \bar{\alpha}k \longrightarrow \bar{\alpha}h \cdot l\beta$$

and so a 2-cell

$$\alpha h \cdot x \cdot l\bar{\beta} : (\alpha \circ \beta) \bullet (\bar{\alpha} \circ \bar{\beta}) \longrightarrow (\alpha \bullet \bar{\alpha}) \circ (\beta \bullet \bar{\beta})$$

i.e. a special cube

$$\chi: \frac{\alpha|\beta}{\bar{\alpha}|\bar{\beta}} \longrightarrow \frac{\alpha}{\bar{\alpha}} \Big| \frac{\beta}{\bar{\beta}}$$

Conditions b(i) and b(ii) of Proposition 5.1.2 say that if either β or $\bar{\alpha}$ is 1, i.e. Id, then χ is equality. Condition b(iii) says that $\delta : \text{Id} \longrightarrow \text{Id}|\text{Id}$ is an equality. μ and τ are also equalities because the hom's are 2-categories.

5.3. Gray categories as intercategories – colax case.

The colax case is similar. Instead of using (b) of Proposition 5.1.2 we use (c) to get a category object in $\mathcal{C}x\mathcal{D}bl$. We again get an intercategory A_c except that now the 1-cells of \mathcal{A} are made into the vertical arrows of A_c , its horizontal arrows being identities. A general cube is now



with a 3-cell $\alpha \longrightarrow \alpha'$ inside. Vertical composition is given by



$$\alpha \bullet \beta = f\beta \cdot \alpha k.$$

Again, δ, μ, τ are equalities but now

$$\chi: \frac{\alpha|\beta}{\bar{\alpha}|\bar{\beta}} \longrightarrow \frac{\alpha}{\bar{\alpha}} \Big| \frac{\beta}{\bar{\beta}}$$

is equality if either α or $\overline{\beta}$ is 1, i.e. id.

5.4. Gray categories as intercategories – symmetric case.

Having two equally good ways of considering true Gray categories as intercategories is a bit unsatisfactory. There is a third better, or more symmetric, way not suggested by Proposition 5.1.2 with quintets as basic cells. Given a true Gray category \mathcal{A} we construct an intercategory A_s as follows.

(1) Objects same as \mathcal{A}

- (2) Transversal arrows are identities
- (3) Horizontal and vertical arrows are 1-cells of \mathcal{A}
- (4) Horizontal and vertical cells are identities
- (5) Basic cells are (op)quintets



(6) Cubes are 3-cells $\alpha \longrightarrow \bar{\alpha}$



Transversal composition is either trivial or, for cubes, composition of 3-cells. Horizontal composition of basic cells is given by

$$A \xrightarrow{f} A' \xrightarrow{f'} A'' \qquad A'' \qquad A \xrightarrow{ff'} A''$$

$$\downarrow k \qquad \downarrow k' \qquad \stackrel{\alpha}{\Rightarrow} \qquad \downarrow k'' \qquad = \qquad k \qquad \downarrow \stackrel{\alpha g' \cdot f\alpha'}{\implies} \quad \downarrow k''$$

$$B \xrightarrow{g} B' \xrightarrow{g'} B'' \qquad B \xrightarrow{gg'} B''$$

$$\alpha \circ \alpha' = (kgg' \xrightarrow{\alpha g'} fk'g' \xrightarrow{f\alpha'} ff'k'')$$

Vertical composition of basic cells is



$$\alpha \bullet \beta = (klh \xrightarrow{k\beta} kgl' \xrightarrow{\alpha l'} fk'l').$$

For cubes, horizontal and vertical composition are given by the same formulas but applied to 3-cells.

Note that there is really no choice for these composites when we work with quintets and they reduce to the ones above when the horizontal or vertical domains and codomains are identities. However, the basic cells are oriented differently in the first case, which is unavoidable.

For interchange on



 $\frac{\alpha | \alpha'}{\beta | \beta'}$ is the top composite and, $\frac{\alpha}{\beta} \left| \frac{\alpha'}{\beta'} \right|$, the bottom in



and

$$\chi: \frac{\alpha |\alpha'}{\beta |\beta'} \longrightarrow \left| \frac{\alpha}{\beta} \right| \frac{\alpha'}{\beta'}$$

is given by the Gray morphism

$$x: kg\beta' \cdot \alpha g'l'' \longrightarrow \alpha l'h' \cdot fk'\beta'$$

in the diamond. If either α or β' is 1, then χ is equality.

We summarize the discussion of this section in the following statement.

5.4.1. THEOREM. True Gray categories can be viewed as intercategories in three ways. They all satisfy the following properties:

- (a) transversal arrows and horizontal and vertical cells are identities,
- (b) all composites are strictly unitary and associative,
- (c) the degenerate interchangers δ, μ, τ are identities, i.e. the intercategory is chiral. The three ways are:
- (1) The lax case
 - (d) vertical arrows are identities,

(e) the interchanger $\chi : \frac{\alpha | \beta}{\bar{\alpha} | \bar{\beta}} \longrightarrow \frac{\alpha}{\bar{\alpha}} \Big| \frac{\beta}{\bar{\beta}}$ is the identity if either β or $\bar{\alpha}$ is Id.

(2) The colax case

(d) horizontal arrows are identities

(e) the interchanger $\chi : \frac{\alpha | \beta}{\bar{\alpha} | \bar{\beta}} \longrightarrow \frac{\alpha}{\bar{\alpha}} | \frac{\beta}{\bar{\beta}}$ is the identity if either α or $\bar{\beta}$ is id.

- (3) The symmetric case
 - (d) there is a bijection between vertical and horizontal arrows



with connecting basic cells



satisfying

(i)

(ii)

$$A = A$$

$$\begin{vmatrix} & & & \\ & & & \\ & & & \\ & & & \\ A \xrightarrow{v_*} & B \\ & & & \\ v & & & \\ & & \epsilon_v \\ & & & \\ B = B \\ & & B \\ & & & & \\ & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & & \\$$

(iii) $(\mathrm{Id}_A)_* = \mathrm{id}_A \text{ and } \epsilon_{\mathrm{Id}_A} = \eta_{\mathrm{Id}_A} = \mathrm{Id}_{\mathrm{id}_A}$ (iv) $(v \bullet w)_* = v_* \circ w_* \text{ and}$

$$\frac{\epsilon_v |\mathrm{Id}_{w_*}}{\mathrm{id}_w | \epsilon_w} = \epsilon_{v \cdot w} \qquad \quad \frac{\eta_v |\mathrm{id}_v}{\mathrm{Id}_{v_*} | \eta_w} = \eta_{v \cdot w}$$

(e) the interchanger $\chi : \frac{\alpha | \beta}{\bar{\alpha} | \bar{\beta}} \longrightarrow \frac{\alpha}{\bar{\alpha}} | \frac{\beta}{\bar{\beta}}$ is the identity if either α or $\bar{\beta}$ is a "commutativity cell".

REMARK: Commutativity cells were introduced in [12], Section 3.1 (called "commutative cells" there). We have to be a bit more careful here because of lax interchange. We say that α is a *commutativity cell* if $f \circ w_* = v_* \circ g$ and

$$A = A \xrightarrow{f} B \xrightarrow{w_*} D$$

$$\left\| \begin{array}{ccc} \eta_v & v \\ \eta_v & v \\ \end{array} \right\| & \left\| \begin{array}{ccc} \alpha & w \\ \phi & \psi \\ \end{array} \right\| \\ A \xrightarrow{v_*} C \xrightarrow{g} D = D \end{array} = D$$

$$Id.$$

It follows easily, using (i) and (iii) above, that

$$A = A \qquad A \xrightarrow{f} B \qquad A = A \qquad A \xrightarrow{v_*} B$$

$$\downarrow v \downarrow id_v \qquad \downarrow v \qquad \parallel Id_f \qquad \sqcup Id_$$

are all commutativity cells.

6. Spans in double categories

6.1. The intercategory of spans in a double category.

Let \mathbb{A} be a (weak) double category with a lax choice of pullbacks, i.e. the diagonal functor

$$\Delta: \mathbb{A} \longrightarrow \mathbb{A}^{\mathbf{P}}$$

has a right adjoint [11], where **P** is the category



In elementary terms, this means the following. Let $\mathbb{A} = \mathbf{A}_2 \xrightarrow[p_2]{m} \mathbf{A}_1 \xrightarrow[d_1]{\rightarrow} \mathbf{A}_0$.

Then \mathbf{A}_0 and \mathbf{A}_1 have pullbacks preserved by ∂_0 and ∂_1 . Furthermore a choice of pullback has been made also preserved by ∂_0 and ∂_1 . So a chosen pullback in \mathbf{A}_1 will look like



The point of choosing pullbacks is not a question of the axiom of choice, which we use unashamedly, but of choosing them compatibly with ∂_0 , ∂_1 . This doesn't follow just from preservation but it is almost always possible, e.g. if every horizontal isomorphism has a companion.

Given such a compatible choice, it follows that $\mathbf{A}_2 = \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1$ has a choice of pullbacks compatible with p_1 and p_2 , i.e. compatible pairs of chosen pullbacks in \mathbf{A}_1 give a pullback in \mathbf{A}_2 . We are not assuming that $m : \mathbf{A}_2 \longrightarrow \mathbf{A}_1$ preserves pullbacks, but there is always a universally given comparison cell



So if we do have a compatible choice of pullbacks it's always lax, as with any right adjoint. The word "lax" in "lax choice of 1-dimensional pullback" is just to emphasize that it is not strong, and "1-dimensional" refers to the 1-dimensional universal property. If id : $\mathbf{A}_0 \longrightarrow \mathbf{A}_1$ preserves pullbacks (not necessarily the chosen ones) we say that A has a lax choice of pullbacks as defined in [10]. This is almost always the case and it is a weak condition to impose. On the other hand, $m : \mathbf{A}_2 \longrightarrow \mathbf{A}_1$ is just as likely to preserve pullbacks as not. If id and m preserve them we say we have a strong choice of pullback.

Assume now that \mathbb{A} has a lax choice of 1-dimensional pullbacks. Let Λ be the category

$$0 \leftarrow 2 \rightarrow 1.$$

Then \mathbb{A}^{Λ} is the double category whose objects are spans of horizontal arrows

$$A_0 \leftarrow A_2 \rightarrow A_1$$

whose horizontal arrows are commutative diagrams of horizontal arrows



whose vertical arrows are spans of cells



and whose double cells are commutative diagrams of cells



That is

$$\mathbb{A}^{\mathbf{\Lambda}} = \mathbf{A}_{2}^{\mathbf{\Lambda}} \implies \mathbf{A}_{1}^{\mathbf{\Lambda}} \implies \mathbf{A}_{0}^{\mathbf{\Lambda}}$$

We have strict double functors $\partial_0, \partial_1 : \mathbb{A}^{\Lambda} \longrightarrow \mathbb{A}$. ∂_0 picks out the 0 part of the diagram, and ∂_1 the 1 part. They are induced by the corresponding functors $\mathbb{1} \xrightarrow[]{\Gamma_1 \cap \mathbb{T}} \Lambda$. We also have a strict double functor id : $\mathbb{A} \longrightarrow \mathbb{A}^{\Lambda}$ coming from $\Lambda \longrightarrow \mathbb{1}$.

The pullback $\mathbb{A}^{\Lambda} \times_{\mathbb{A}} \mathbb{A}^{\Lambda}$ is $\mathbb{A}^{\mathbf{M}}$ where **M** is the category

$$0 \leftarrow 3 \rightarrow 1 \leftarrow 4 \rightarrow 2.$$

The pullback (lax) functor $\mathbb{A}^{\mathbf{P}} \longrightarrow \mathbb{A}$ induces a lax functor $m : \mathbb{A}^{\mathbf{M}} \longrightarrow \mathbb{A}^{\mathbf{\Lambda}}$ and produces a pseudocategory

$$\mathbb{A}^{\mathbf{M}} \xrightarrow[p_2]{p_1} \mathbb{A}^{\mathbf{\Lambda}} \xrightarrow[]{\partial_0} \\[1ex] \xrightarrow[p_2]{\leftarrow} \mathrm{id} \\[1ex] \xrightarrow[\partial_1]{\rightarrow}} \mathbb{A}$$

in $\mathcal{L}x\mathcal{D}bl$. In this way we get an intercategory, that we call $\mathsf{Span}(\mathbb{A})$. As a double pseudocategory in \mathcal{CAT} , it looks like



whose rows are the double categories $\text{Span}\mathbf{A}_0$, $\text{Span}\mathbf{A}_1$ and $\text{Span}\mathbf{A}_2$ of [6]. Recall from there that an arbitrary functor $F: \mathbf{B} \longrightarrow \mathbf{C}$ between categories with pullbacks induces a colax normal functor

$$\operatorname{Span} F : \operatorname{Span} \mathbf{B} \longrightarrow \operatorname{Span} \mathbf{C}.$$

If F preserves pullbacks, then SpanF is a strong functor. To make $\text{Span}\mathbf{B}$ and $\text{Span}\mathbf{C}$ into double categories, a choice of pullback must be made (to define vertical composition). If F preserves these choices, then SpanF is a strict functor.

In this way we get the alternative description of $\mathsf{Span}(\mathbb{A})$ as a pseudocategory object

$$\operatorname{Span} \mathbf{A}_2 \longrightarrow \operatorname{Span} \mathbf{A}_1 \longrightarrow \operatorname{Span} \mathbf{A}_0$$

in $\mathcal{C}x\mathcal{D}bl$.

Referring to the table of Section 4 of [13] we get a more detailed description of $\mathsf{Span}(\mathbb{A})$. Its

(1) objects are those of \mathbb{A} ,

(2) transversal arrows are the horizontal morphisms of \mathbb{A} ,

(3) horizontal arrows are spans of horizontal morphisms of \mathbb{A} ,

- (4) vertical arrows are the vertical morphisms of \mathbb{A} ,
- (5) horizontal cells are commutative diagrams as in (a) above,
- (6) vertical cells are the double cells of \mathbb{A} ,
- (7) basic cells are spans of double cells as in (b) above,
- (8) cubes are commutative diagrams of double cells as in (c).

Compositions are obvious, either coming from \mathbb{A} or the composition of spans \otimes . To see how interchange works, consider the following diagram in \mathbb{A} :



To calculate $\frac{\alpha|\beta}{\bar{\alpha}|\bar{\beta}}$ we take the pullbacks



and then compose the left half and right halves of



whereas to calculate $\frac{\alpha}{\bar{\alpha}} \left| \frac{\beta}{\bar{\beta}} \right|$ we take the pullback





The comparison

$$\gamma: (v_1 \times_{v_2} w_1) \bullet (x_1 \times_{x_2} y_1) \longrightarrow (v_1 \bullet x_1) \times_{v_2 \bullet x_2} (w_1 \bullet y_1)$$

gives a morphism of spans in \mathbf{A}_2 which is

$$\chi: \frac{\alpha|\beta}{\bar{\alpha}|\bar{\beta}} \longrightarrow \frac{\alpha}{\bar{\alpha}} \Big| \frac{\beta}{\bar{\beta}}.$$

For the degenerate interchangers μ, δ, τ we have the following. The horizontal identity id_v is



and $\mathrm{id}_v \bullet \mathrm{id}_{\bar{v}} = \mathrm{id}_{v \bullet \bar{v}}$ so μ is the identity. The vertical identity Id_f is

$$\begin{array}{c|c} A_{0} & \xleftarrow{f_{o}} & A_{2} \xrightarrow{f_{1}} & A_{1} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \operatorname{id}_{A_{0}} & \downarrow & \operatorname{id}_{f_{0}} & \downarrow & \operatorname{id}_{A_{2}} & \operatorname{id}_{f_{1}} & \downarrow & \operatorname{id}_{A_{2}} \\ & A_{0} & \xleftarrow{f_{0}} & A_{2} \xrightarrow{f_{1}} & A_{1} \end{array}$$

and horizontal composition of two of these is done by taking the pullback of id_{f_1} with id_{g_0} . If pullbacks in A are not normal we get a nontrivial comparison

$$\operatorname{id}_{A_2 \times_{A_1} B_2} \longrightarrow \operatorname{id}_{A_2} \times_{\operatorname{id}_{A_1}} \operatorname{id}_{g_2}$$

which gives a nontrivial

$$\delta: \mathrm{Id}_{f\otimes g} \longrightarrow \mathrm{Id}_f \otimes \mathrm{Id}_g.$$

Finally id_{Id_A} is



and Id_{id} is



so they are equal and $\tau : \mathrm{Id}_{\mathrm{id}_A} \longrightarrow \mathrm{id}_{\mathrm{Id}_A}$ is the identity.

6.2. DOUBLE SPANS.

We now consider some specific examples. The first is the double category SpanA for a category with pullbacks. It has a strong choice of pullbacks. SpanSpanA is an important construction and deserves a special name Span^2A . A general cube looks like



the front and back, the basic cells, being spans of spans. All the interchangers are isomorphisms.

If **B** is another category with pullbacks and $F : \mathbf{A} \longrightarrow \mathbf{B}$ an arbitrary functor, then we get a colax-colax functor $\mathbf{Span}^2 F : \mathbf{Span}^2 \mathbf{A} \longrightarrow \mathbf{Span}^2 \mathbf{B}$. If $U : \mathbf{B} \longrightarrow \mathbf{A}$ is right adjoint to F, then U preserves pullbacks and induces a strong-strong functor $\mathbf{Span}^2 U :$ $\mathbf{Span}^2 \mathbf{B} \longrightarrow \mathbf{Span}^2 \mathbf{A}$. We can consider it as a colax-lax functor and as such it is a conjoint to $\mathbf{Span}^2 F$ in ICat.

6.3. MATRICES IN A MONOIDAL CATEGORY REVISITED.

A related example is the intercategory $\mathsf{SM}(\mathbf{V})$ of Section 2. For a monoidal category (\mathbf{V}, \otimes, I) with coproducts over which \otimes distributes, we have the double category \mathbf{V} -Set, introduced in [19], of sets, functions and \mathbf{V} -matrices. If \mathbf{V} has pullbacks, then \mathbf{V} -Set has a lax choice of pullbacks and $\mathsf{Span}(\mathbf{V}$ -Set) is $\mathsf{SM}(\mathbf{V})$.

6.4. Spans of cospans.

An equally interesting example is the following. Let \mathbf{A} be a category with pullbacks and pushouts. Then the double category of cospans $\mathbb{C}osp\mathbf{A}$ has a lax (normal) choice of pullbacks. This gives an intercategory Span $\mathbb{C}osp\mathbf{A}$ which we will call Span $\mathbb{C}osp\mathbf{A}$. A basic cell is a span of cospans and a general cube is a commutative diagram



Transversal composition is just composition in **A**, horizontal composition is span composition and given by pullback, and vertical composition is cospan composition given by pushout. The χ is almost never an isomorphism but the other interchangers μ, δ, τ always are. More details can be found in [14].

SpanCospA is closely related to the product-coproduct duoidal category. Let A be the intercategory obtained from the duoidal category $(\mathbf{A}, \times, 1, +, 0)$ as in Section 2. There are two canonical inclusions of A into SpanCospA. F_0 which takes a basic cell of A



to the basic cell

and F_1 which takes it to

 F_0 is strong-lax and F_1 is colax-strong. There is also a canonical morphism G: SpanCospA \longrightarrow A which picks out the middle object of a span of cospans



Although GF_0 and GF_1 are the identity on A, there is no conjointness or adjointness relationship between the F's and G, contrary to the situation with $SM(\mathbf{V})$ of Section 2.

By duality we can start with a double category \mathbb{A} with a colax choice of pushouts and construct an intercategory of cospans. Because the interchanger χ has a given direction, dualization in the transversal direction forces the switching of horizontal and vertical. So CospA has cospans of horizontal arrows of A as its vertical arrows. The transversal arrows of CospA are the horizontal morphisms of A and the horizontal arrows of CospA are the vertical arrows of A.

If **A** is a category with pushouts and pullbacks then we can form $\mathsf{Cosp}(\mathsf{Span}\mathbf{A})$ and it is exactly the same as $\mathsf{Span}(\mathbb{C}\mathsf{osp}\mathbf{A})$, i.e. they are both $\mathsf{Span}\mathsf{Cosp}\mathbf{A}$. Cherubini, Sabadini and Walters have used $\mathsf{Span}\mathsf{Cosp}\mathbf{A}$ in their work on concurrent systems [5]. They use the category of graphs, $\mathsf{Set}^{:,:}$, as **A**.

If A is a category with pushouts, we can form the intercategory of *double cospans*, $Cosp^{2}(A) = Cosp(CospA)$. Double cospans were introduced by Morton in the context of quantum field theory, first as an arXiv preprint in 2006 and later published as [18]. They were presented as Verity double bicategories so there were no transversal arrows. This was taken up in [9], where higher cospans (and spans) were introduced, including their transversal morphisms, which are important for us here.

6.5. Profunctors and spans in Cat.

An interesting example of a double category with a lax choice of pullbacks is Cat, the double category whose objects are small categories, horizontal arrows functors and vertical arrows profunctors. We choose the usual construction for pullbacks in **Cat**, viz. pairs of objects and pairs of arrows. Given double cells in Cat



we take $R \times_P S : \mathbf{B} \times_{\mathbf{A}} \mathbf{C} \longrightarrow \bar{\mathbf{B}} \times_{\bar{\mathbf{A}}} \bar{\mathbf{C}}$ to be

$$R \times_P S((B,C), (\bar{B}, \bar{C})) = \{(x, y) | x \in R(B, \bar{B}), y \in S(c, c'), \rho(x) = \sigma(y)\}$$

We can represent such an element as

$$(x,y):(B,C)\longrightarrow(\bar{B},\bar{C})$$

where $x: B \xrightarrow{\bullet}_{R} \bar{B}$ and $y: C \xrightarrow{\bullet}_{S} \bar{C}$ and

$$\left(\rho(x): FB \xrightarrow{P} \bar{F}\bar{B}\right) = \left(\sigma(y): GC \xrightarrow{P} \bar{G}\bar{C}\right).$$

The identities are the hom functors and from the definition of morphisms in the pullbacks we see that $id_{\mathbf{B}} \times_{id_{\mathbf{A}}} id_{\mathbf{C}} = id_{\mathbf{B} \times_{\mathbf{A}} \mathbf{C}}$, i.e. we have a unitary choice of pullback.

On the other hand pullbacks are not strong but merely lax (normal) as the following example shows:



A profunctor $1 \longrightarrow 1$ is just a set and $R \otimes \overline{R} = R \times \overline{R}$, $S \otimes \overline{S} = S \times \overline{S}$. $1: 1 \longrightarrow 2$ and $1: 2 \longrightarrow 1$ are the constant profunctors with value 1 and $1 \otimes 1 = 1$. So $(R \otimes \overline{R}) \times_{1 \otimes 1}$ $(S \otimes \overline{S}) = R \times \overline{R} \times S \times \overline{S}$. But $(R \times_1 S) \otimes (\overline{R} \times_1 \overline{S})$ is the composite

$$\mathbb{1} \longrightarrow 0 \longrightarrow \mathbb{1}$$

which is 0.

Thus the intercategory $\mathsf{Span}(\mathbb{C}at)$ has a non invertible interchanger χ .

6.6. GENERALIZED SPANS AND COSPANS.

We introduce here a class of intercategories, parametrized by quintets, which give nice examples in which the interchangers are not isomorphisms. This generalizes the spans of cospans of Section 6.4.

Consider a quintet in Cat



in which **B** and **D** have pullbacks, and **C** and **D** have pushouts. (No preservation properties are assumed on G or K.)

We construct an intercategory SC_{ϕ} as follows. A basic cell consists of 9 objects and 12 morphisms as in



 $A_{21} \quad FA_{21} \longleftarrow B_2 \longrightarrow FA_{22} \quad A_{22}$

satisfying four commutativities (i, j = 1, 2)



From this one may infer that the objects of SC_{ϕ} are the objects of \mathbf{A} , and its horizontal arrows are F-spans in \mathbf{B} , i.e. spans whose domain and codomain are F of something in \mathbf{A} . Similarly, the vertical arrows are H-cospans in \mathbf{C} . The transversal arrows are natural families of morphisms, i.e. $a_{ij} : A_{ij} \longrightarrow A'_{ij}, b_i : B_i \longrightarrow B'_i, c_j : C_j \longrightarrow C'_j, d : D \longrightarrow D'$ commuting with the structural morphisms. When F, G, H, K, ϕ are all identities, this is just the SpanCospA of 6.4.

Horizontal composition is span composition, i.e. given by pulling back:

The vertical arrows in the middle are pullbacks induced by



and are the composites

$$K(B_i \times_{FA_{i2}} B'_i) \longrightarrow KB_i \times_{KFA_{i2}} KB'_i \longrightarrow KB_i \times_{GHA_{i2}} KB'_i \longrightarrow D \times_{GC_2} D'.$$

Vertical composition is dual.

The horizontal identities id_A and id_C are shown in the diagram



where the vertical arrows in the middle are

$$KFA \xrightarrow{\phi A} GHA \longrightarrow GC$$
 and $KF\bar{A} \xrightarrow{\phi A} GH\bar{A} \longrightarrow GC$.

The vertical identities, Id_A and Id_B , which are dual, are displayed in

$$A \quad FA \longleftarrow B \longrightarrow FA' \quad A'$$

$$HA \quad KB \qquad HA'$$

$$\downarrow 1 \qquad \downarrow 1 \qquad \downarrow 1$$

$$HA \quad GHA \longleftarrow KB \longrightarrow GHA' \quad HA'$$

$$\uparrow 1 \qquad \uparrow 1 \qquad \uparrow 1$$

$$HA \qquad KB \qquad HA'$$

$$A \quad FA \longleftarrow B \longrightarrow FA' \quad A'$$

where the horizontal arrows in the middle are

$$KB \longrightarrow KFA \xrightarrow{\phi A} GHA \text{ and } KB \longrightarrow KFA' \xrightarrow{\phi A'} GHA'$$

By abuse of notation we write $id_A = FA$, $id_C = GC$, $Id_A = HA$, and $Id_B = KB$. Then $Id_{id_A} = Id_{FA} = KFA$ and $id_{Id_A} = id_{HA} = GHA$, and $\tau A : Id_{id_A} \longrightarrow id_{Id_A}$ is exactly

$$\phi A: KFA \longrightarrow GHA \tag{1}$$

For a composable pair of F-spans B and B' we have $B|B' = B \times_{FA'} B'$ so $\mathrm{Id}_{B|B'} = K(B \times_{FA'} B')$ whereas $\mathrm{Id}_B|\mathrm{Id}_{B'}$ is $KB \times_{GHA'} KB'$ and $\delta(B, B') : \mathrm{Id}_{B|B'} \longrightarrow \mathrm{Id}_B|\mathrm{Id}_{B'}$ is given by

$$K(B \times_{FA'} B') \xrightarrow{\lambda} KB \times_{KFA'} KB' \xrightarrow{1 \times_{\phi A'} 1} KB \times_{GHA'} KB' \tag{2}$$

The μ is dual to this. $\mu(C, \overline{C}) : \frac{\mathrm{id}_C}{\mathrm{id}_{\overline{C}}} \longrightarrow \mathrm{id}_{\frac{C}{\overline{C}}}$ is the composite

$$GC +_{KF\bar{A}} G\bar{C} \xrightarrow{1+_{\phi A}1} GC +_{GH\bar{A}} G\bar{C} \xrightarrow{\gamma} G(C +_{H\bar{A}} \bar{C})$$
(3)

The formula for χ is similar but more complicated notationally. Consider four basic cells (i, j = 1, 2)

$$A_{ij} \xrightarrow{B_{ij}} A_{i,j+1}$$

$$C_{ij} \downarrow \qquad D_{ij} \qquad \downarrow^{C_{i,j+1}}$$

$$A_{i+1,j} \xrightarrow{B_{i+1,j}} A_{i+1,j+1}$$

with structure morphisms as above. Then $(D_{11} \circ D_{12}) \bullet (D_{21} \circ D_{22})$ is the pushout P of



whereas $(D_{11} \bullet D_{21}) \circ (D_{12} \bullet D_{22})$ is the pullback Q of



A morphism from a pushout to a pullback is given by four morphisms satisfying four equations. The ones giving χ are:

$$D_{i1} \times_{GC_{i2}} D_{i2} \xrightarrow{proj_j} D_{ij} \xrightarrow{inj_i} D_{1j} +_{KB_{2j}} D_{2j}$$

It is difficult to determine when such a χ is our isomorphism. Some insight can be gleaned by decomposing χ into a composite as follows. Consider the diagram, which is a span of cospans



We can take the pullback of each row and then the pushout of the resulting diagram or the other way around and we get the canonical lim-colim comparison morphism θ from the pushout P' of



to the pullback Q' of



These are almost the right things but the pushout is over the wrong object, and similarly for the pullback. We have the canonical comparison

$$\kappa: K(B_{21} \times_{FA_{22}} B_{22}) \longrightarrow KB_{21} \times_{KFA_{22}} KB_{22}$$

which induces an epimorphism:

$$1 +_{\kappa} 1 : P \longrightarrow P'$$

Similarly we have

$$GC_{12} +_{KFA_{22}} GC_{22} \xrightarrow{1+_{\phi A_{22}} 1} GC_{12} +_{GHA_{22}} GC_{22} \xrightarrow{\gamma} G(C_{12} +_{HA_{22}} C_{22})$$

inducing morphisms

$$Q' \xrightarrow{1 \times_{\psi} 1} Q'' \xrightarrow{1 \times_{\gamma} 1} Q$$

where $\psi = 1 + \phi A_{22} 1$ and Q'' is the pullback over the middle object. Then

$$\chi = \left(P \xrightarrow{1+\kappa 1} P' \xrightarrow{\theta} Q' \xrightarrow{1\times\psi 1} Q'' \xrightarrow{1\times\gamma 1} Q\right) \tag{4}$$

6.6.1. THEOREM. In SC_{ϕ} we have:

- (1) τ is an isomorphism if and only if ϕ is;
- (2) δ is an isomorphism if and only if K preserves F-pullbacks and ϕ is a monomorphism;
- (3) μ is an isomorphism if and only if G preserves H-pushouts and ϕ is an epimorphism;
- (4) χ is an isomorphism if and only if
 - (a) ϕ is an isomorphism,
 - (b) K transforms F-pullbacks into quasi-pullbacks,
 - (c) G transforms H-pushouts into quasi-pushouts,
 - (d) G-pullbacks commute with K-pushouts.

PROOF. (1) $\tau A = \phi A$ so (1) is obvious.

(2) δ is given in formula (2) above as $(1 \times_{\phi A'} 1)\lambda$, and $1 \times_{\phi A'} 1$ is an epimorphism, so δ is an isomorphism if and only if both λ and $1 \times_{\phi A'} 1$ are. That λ is an isomorphism is the definition of K preserving F-pullbacks.

Consider

$$KB \xrightarrow{1_{KB}} KB$$

$$KFA' \xrightarrow{\phi A'} GHA'$$

$$KB' \xrightarrow{1_{KB'}} KB'$$

 $1 \times_{\phi A'} 1$ is the morphism this induces from the pullback of the three objects on the left to the pullback of the other three. So, if $\phi A'$ is monic then $1 \times_{\phi A'} 1$ is an isomorphism. Conversely, we can take $B \longrightarrow FA'$ and $B' \longrightarrow FA'$ to be the identity $1_{FA'}$. Then $1 \times_{\phi A'} 1$ is the diagonal for the kernel pair of $\phi A'$ and that's invertible if and only if $\phi A'$ is monic.

(3) This is the dual to (2).

(4) χ is given by (4) above in which $1 +_{\kappa} 1$ is an epimorphism and $1 \times_{\psi} 1$ and $1 \times_{\gamma} 1$ are monomorphisms, so χ is invertible if and only each of these morphisms as well as θ are.

Suppose χ is invertible. τ is a special case of χ , so $\phi = \tau$ is invertible, which gives (a).

To say that K transforms F-pullbacks into quasi-pullbacks means that the canonical morphism

$$\kappa: K(B \times_{FA} B') \longrightarrow KB \times_{KFA} KB'$$

is an epimorphism, for every diagram



Denote the *F*-span $FA \stackrel{b}{\longleftrightarrow} B \stackrel{b}{\longrightarrow} FA$ by *B* and similarly for *B'*. If we evaluate χ at $(Id_B, Id_{B'}; Id_B, Id_{B'})$, then the legs of the diagram whose pushout is *P'* are both equal to $1 \times_{\phi A} 1$, which is an isomorphism. Thus $1 \times_{\kappa} 1 : P \longrightarrow P'$ is the codiagonal for the cokernel pair of κ , and as $1 \times_{\kappa} 1$ is invertible, κ is an epimorphism. This proves (b).

(c) is dual to (b).

By (d) we mean that the morphism θ is invertible, so this holds by definition.

From the above, it is clear that if (a)-(d) hold, then χ is invertible.

6.6.2. REMARK. Condition 4(d) is clearly unsatisfactory as it is. We don't know how to formulate it in a nicer, more general way. It is certainly worth further investigation.

This theorem gives many examples of intercategories in which none of the interchangers are invertible. Any quintet in which the ϕ is neither monic nor epic gives one, for example.

There is a similar class of examples generalizing the double spans of Section 6.2. It starts again with a quintet



but now **B**, **C**, **D** are required to have pullbacks. The details are more complicated and will appear elsewhere.

7. The intercategory **Set**

7.1. INTERMONADS.

As is well known, a small category is a monad in the bicategory of spans, or better, a vertical monad in the double category Set of sets, functions and spans. Better because it is here that functors appear naturally. So a small category corresponds to a lax functor $1 \longrightarrow Set$.

Examining the notion of small double category, which has two kinds of morphisms, cells and various domains and codomains, we see a span of spans. We wish to code up the compositions and identities as a sort of double monad. This will live in the intercategory Span(Span(Set)) = Span(Set) which we call the intercategory of sets and denote Set. A small double category will then turn out to be a lax-lax morphism of intercategories $1 \longrightarrow Set$.

Let us examine the structure of a lax-lax functor from $1\!\!1$ to an arbitrary intercategory A. The unique basic cell of $1\!\!1$ gives



t is a horizontal monad whose structure is given by special horizontal cells



composed in the transversal direction. T is a vertical monad whose structure is given by vertical cells, U and M, also composed in the transversal direction.

The main structure is on D. It is a horizontal and vertical monad whose structural morphisms are cubes



These must satisfy the following conditions.

(1) (Horizontal monad)



(2) (Vertical monad)







(3) (Horizontal/vertical compatibility)



7.1.1. DEFINITION. We call (D, u, m, U, M) as above satisfying conditions (1)-(3) an *intermonad* in A.

Not surprisingly, an intermonad in Set is a small (strict) double category. Conditions (3) express the interchange law. More generally, for a category A with pullbacks, an intermonad in $\text{Span}^2 A$ is a double category object in A.

If \mathbf{V} is a duoidal category, considered as an intercategory as in Section 2, then an intermonad in \mathbf{V} is what Aguiar and Mahajan [1] call a double monoid.

If \mathcal{A} is a bicategory, then an intermonad in the intercategory $Q(\mathcal{A})$ of quintets, as introduced in Section 4, reduces to a pair of monads with a distributive law between them.

7.2. Hom functors for intercategories.

We end with an example of morphism of intercategories reinforcing the idea that $Span^{2}(Set)$ is really the intercategory of sets. Let A be an intercategory and X a fixed object of A. Define the hom functor $H : A \longrightarrow Set$ as follows.

(1) H(A) is the set of transversal arrows $X \longrightarrow A$.

(2) For a transversal arrow $f : A \longrightarrow A'$, $H(f) : H(A) \longrightarrow H(A')$ is defined by composing with f as usual. This is a function, so a transversal arrow of Set, and is strictly functorial in f.

(3) For a horizontal arrow $h: A \longrightarrow B$, H(h) is the span



H(h) is the set of all horizontal cells



with $p_0(\phi) = f$ and $p_1(\phi) = g$. We consider H(h) as a horizontal arrow in Set. (4) For vertical arrows $v : A \longrightarrow \overline{A}$ we define H(v) to be the span of all vertical cells



now considered as a vertical arrow of Set.

(5) The action of H on horizontal (resp. vertical) cells is supposed to be a morphism of spans and is given by transversal composition. This is strictly functorial as it should be.

So far this is just like the hom functors for double categories in [19]. In particular H will be lax on horizontal (resp. vertical) arrows.

However some care is needed in the definition of H on basic cells



It will be a span of spans



 $H(\alpha)$ will be the set of cubes



But there is a choice for the (hidden) back face, either Id_{id_x} or id_{Id_x} . Only Id_{id_x} works and we'll see why below.

(6) $\mathsf{H}(\alpha)$ is the set of cubes $c : \mathrm{Id}_{\mathrm{id}_x} \longrightarrow \alpha$ with the projections $s_0(c) = \psi$, $s_1(c) = \theta$ (the right face of c), $t_0(c) = \phi$, $t_1(c) = \overline{\phi}$.

(7) The horizontal laxity morphisms of H are as follows:

$$\mathsf{H}_h: \mathrm{id}_{\mathsf{H}}(v) \longrightarrow \mathsf{H}(\mathrm{id}_v)$$

$$(\psi: \mathrm{Id}_X \longrightarrow v) \longmapsto (\mathrm{Id}_{\mathrm{id}_X} \xrightarrow{\tau} \mathrm{id}_{\mathrm{Id}_X} \xrightarrow{\mathrm{id}_{\psi}} \mathrm{id}_v)$$

and

$$\mathsf{H}_h: \mathsf{H}(\alpha)|\mathsf{H}(\beta) \longrightarrow \mathsf{H}(\alpha|\beta)$$

$$(c: \mathrm{Id}_{\mathrm{id}_X} \longrightarrow \alpha, d: \mathrm{Id}_{\mathrm{id}_X} \longrightarrow \beta) \longmapsto (\mathrm{Id}_{\mathrm{id}_X} \xrightarrow{\mathrm{Id}_{\lambda^{-1}}} \mathrm{Id}_{\mathrm{id}_X|\mathrm{id}_X} \xrightarrow{\delta} \mathrm{Id}_{\mathrm{id}_X} |\mathrm{Id}_{\mathrm{id}_X} \xrightarrow{c|d} \alpha|\beta)$$

(8) The vertical laxity morphisms are as follows.

$$\mathsf{H}_v: \mathrm{Id}_{\mathsf{H}(h)} \longrightarrow \mathsf{H}(\mathrm{Id}_h)$$

$$\begin{aligned} (\phi: \mathrm{id}_X \longrightarrow h) \longmapsto (\mathrm{Id}_{\mathrm{id}_X} \xrightarrow{\mathrm{Id}_\phi} \mathrm{Id}_h) \\ \mathsf{H}_v: \frac{\mathsf{H}(\alpha)}{\mathsf{H}(\bar{\alpha})} \longrightarrow \mathsf{H}(\frac{\alpha}{\bar{\alpha}}) \\ (c: \mathrm{Id}_{\mathrm{id}_X} \longrightarrow \alpha, \bar{c}: \mathrm{Id}_{\mathrm{id}_X} \longrightarrow \bar{\alpha}) \longmapsto \left(\mathrm{Id}_{\mathrm{id}_X} \xrightarrow{\lambda^{-1}} \frac{\mathrm{Id}_{\mathrm{id}_X}}{\mathrm{Id}_{\mathrm{id}_X}} \xrightarrow{\frac{c}{\bar{c}}} \frac{\alpha}{\bar{\alpha}} \right). \end{aligned}$$

This completes the description of H. Note that in (7) we had to use τ and δ whereas in (8) we only used the structural isomorphisms of A. Had we defined H using id_{Id_X} as domain, (7) would only use the structural isomorphism whereas (8) would need τ and μ , both of which go in the wrong direction.

H has to satisfy a number of conditions, namely (5)-(14) of Section 5, [13]. This is merely a question of working through the definitions above in the context of (5)-(14) and using the coherence conditions of Section 4 in [13]. We do a few representative examples in detail.

In all of these diagrams, the objects are spans of spans of sets and the arrows are morphisms of such. In order to show commutativity it is sufficient to take an element of the middle set and follow its paths around the diagram and verify that we get the same thing in both cases.

Let's take (5) for example. For a basic cell



we have to verify commutativity of



The upper left corner is the span of spans



and an element of $\mathsf{H}(h) \times_{\mathsf{H}(h)} \mathsf{H}(\alpha)$ is a pair consisting of a horizontal cell $\phi : \mathrm{id}_X \longrightarrow h$ and a cube $c : \mathrm{Id}_{\mathrm{id}_X} \longrightarrow \alpha$ whose vertical domain is ϕ



The λ' on the left is the isomorphism which takes (ϕ, c) to c. Going around the square gives



Naturality of λ'



gives $\lambda' \left(\frac{\mathrm{Id}_{\phi}}{c}\right) \lambda'^{-1} = c.$

Conditions (6) and (7) are virtually the same, using only the vertical double category coherence.

Condition (8) is the transpose of (5), but because it is about horizontal composition it will involve τ and δ and intercategory coherence. We must verify commutativity of



An element of the top left corner is a pair consisting of a vertical cell $\psi : \operatorname{Id}_X \longrightarrow v$ and a cube $c : \operatorname{Id}_{\operatorname{id}_X} \longrightarrow \alpha$ whose horizontal domain is ψ



 λ takes (ψ, c) to c, whereas going around the square we get first of all $((\mathrm{id}_{\psi})\tau, c)$, then $((\mathrm{id}_{\psi})(\tau)|c) \cdot \delta \mathrm{Id}_{\lambda^{-1}}$ and finally we multiply by λ to get the long way around the diagram



The top square is condition (30) from Section 4 of [13]. Conditions (9) and (10) are very much the same. Condition (11) reduces to naturality of τ . For condition (12) we must verify the commutativity of



This reduces to checking that the following diagram commutes for any two vertically composable vertical cells $\psi : \operatorname{Id}_X \longrightarrow v$ and $\overline{\psi} : \operatorname{Id}_X \longrightarrow \overline{v}$.



where the middle span is (22) of Section 4 in [13] and the other two squares are naturality. Conditions (13) and (14) are similar and left to the reader.

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