Intercategories

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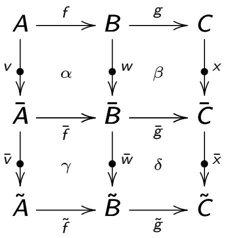
Introduction

- A study of the interchange law
- Intercategory (short for interchange category)
- Kind of triple category
  - Has three kinds of arrows
  - Three kinds of 2-dimensional cells
  - Triple cells (cubes)
- Not a generalization of tricategory
  - One composition is strictly associative and unitary
  - Other two up to isomorphism (with bicategorical type coherence)
  - Interchange is lax

Double Categories

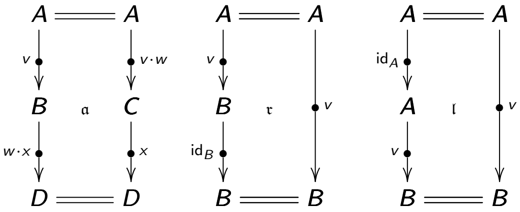
Category object $\mathbb{A}$ in **Cat**: $\mathbf{A}_2 \begin{smallmatrix}\longrightarrow \\ \longrightarrow \\ \longrightarrow\end{smallmatrix} \mathbf{A}_1 \begin{smallmatrix}\longrightarrow \\ \longleftarrow \\ \longrightarrow\end{smallmatrix} \mathbf{A}_0$

- Objects of $\mathbf{A}_0$ are objects of $\mathbb{A}$
- Morphisms of $\mathbf{A}_0$ are horizontal arrows
- Objects of $\mathbf{A}_1$ are vertical arrows
- Morphisms of $\mathbf{A}_1$ are double cells
- Interchange $\dfrac{\alpha|\beta}{\gamma|\delta} = \dfrac{\alpha}{\gamma}\Big| \dfrac{\beta}{\delta}$

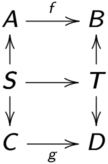


Weak Double Categories

In a *weak* double category, we allow vertical composition to be associative and unitary up to special isomorphism

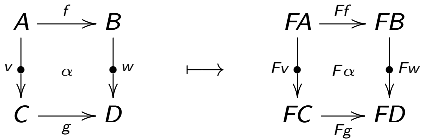


satisfying the usual coherence conditions (pentagon, etc.)
EXAMPLE: $\mathbb{S}$pan**A** for **A** a category with pullbacks
A double cell is

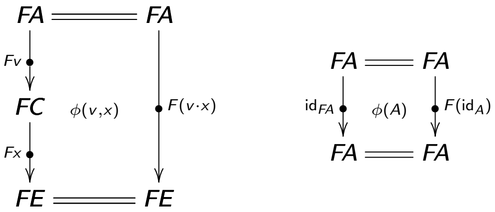


Morphisms

A *lax morphism* $F : \mathbb{A} \longrightarrow \mathbb{X}$



horizontally functorial, but vertically we are given special cells



horizontally natural and satisfying associativity and unitary laws
A *colax morphism*, ϕ's go in opposite direction
EXAMPLE: $\mathbb{S}\mathrm{pan}(F) : \mathbb{S}\mathrm{pan}\mathbf{A} \longrightarrow \mathbb{S}\mathrm{pan}\mathbf{B}$

Theorem

There is a strict double category $\mathbb{D}\mathrm{bl}$ *whose objects are (small) weak double categories, whose horizontal arrows are lax functors, whose vertical arrows are colax functors and whose double cells are horizontal transformations:*

$$
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{F} & \mathbb{X} \\
{\scriptstyle V}\big\downarrow & \pi & \big\downarrow{\scriptstyle W} \\
\mathbb{B} & \xrightarrow[G]{} & \mathbb{Y}
\end{array}
$$

$$
\begin{array}{c}
A \\
{\scriptstyle u}\big\downarrow \\
\bar{A}
\end{array}
\quad \longmapsto \quad
\begin{array}{ccc}
WFA & \xrightarrow{\pi A} & GVA \\
{\scriptstyle WFu}\big\downarrow & \pi u & \big\downarrow{\scriptstyle GVu} \\
WF\bar{A} & \xrightarrow[\pi \bar{A}]{} & GV\bar{A}
\end{array}
$$

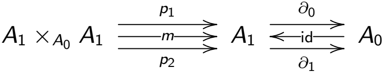
horizontally natural, vertically functorial

NOTE: For 2-categories considered as horizontal double categories π is a 2-natural transformation. For bicategories considered as vertical double categories π is a co-icon (Lack)

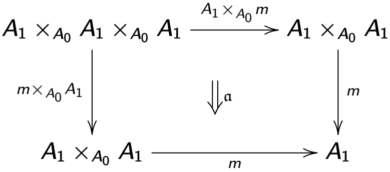
- By taking horizontal (vertical) arrows and special cells we get 2-categories $\mathcal{D}bl\mathcal{L}ax$ and $\mathcal{D}bl\mathcal{C}olax$

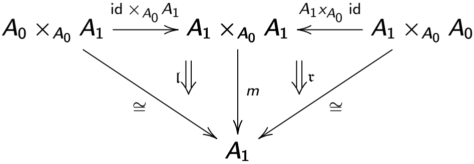
Pseudocategories

In 2-categories with pullbacks we can weaken the associativity and unitary laws for category objects



to giving coherent isomorphisms





A weak double category is a pseudocategory in $\mathbb{C}\mathrm{at}$

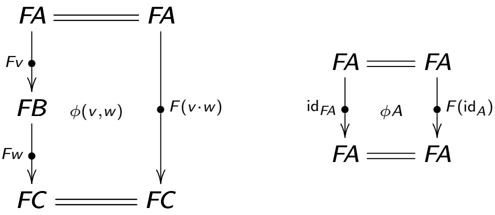
- We only assume that the iterated pullbacks $A_1 \times_{A_0} A_1 \ldots \times_{A_0} A_1$ exist
- We have lax and colax morphisms of pseudocategories and horizontal transformations as above

Theorem

For any 2-category $\mathcal{A}$ we get a strict double category $\mathbb{P}\mathrm{s}\mathbb{C}\mathrm{at}(\mathcal{A})$ whose objects are pseudocategories in $\mathcal{A}$, horizontal arrows are lax morphisms, vertical arrows are colax morphisms, and double cells horizontal transformations

Strict Double Functors

A lax functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ is *strict* if the laxity cells



are *identities*
This means, not only does F preserve vertical composition on the nose, but also the structural isomorphisms $\mathfrak{a}, \mathfrak{l}, \mathfrak{r}$

Proposition

The set theoretical pullback of strict double functors is a weak double category and the projections are strict. It is a 2-pullback in either of the 2-categories, $\mathcal{D}bl\mathcal{L}ax$ *or* $\mathcal{D}bl\mathcal{C}olax$

Intercategories

Definition

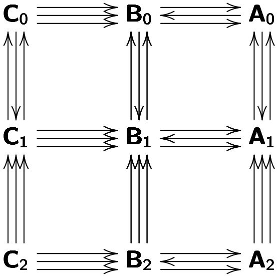
An *intercategory* is a pseudocategory

$$\mathbb{C} \underset{p_2}{\overset{p_1}{\underset{m}{\rightrightarrows\!\!\to}}} \mathbb{B} \underset{\partial_1}{\overset{\partial_0}{\underset{\text{id}}{\rightleftarrows\!\!\to}}} \mathbb{A}$$

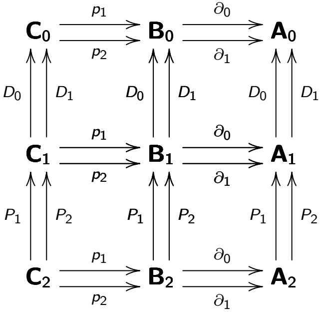
in $\mathcal{D}bl\mathcal{L}ax$ with ∂_0 and ∂_1 *strict* morphisms

- The lax and colax morphisms of pseudocategories give two kinds of morphism of intercategory, lax-lax and colax-lax, which form part of a strict double category $\mathbb{ICat}$
- Why lax?
  An intercategory can equally well be defined as a pseudocategory in $\mathcal{D}bl\mathcal{C}olax$
  $$\mathbb{X}_2 \ \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\ \mathbb{X}_1 \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\ \mathbb{X}_0$$
  But the equivalence is not completely straightforward
  The morphisms are not the same: we still get colax-lax but a new one, colax-colax
  We get another strict double category $\mathbb{ICat}^*$

3 × 3 Diagram of Categories



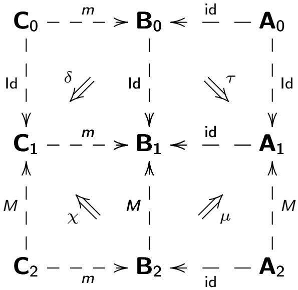
3×3 Diagram of Categories



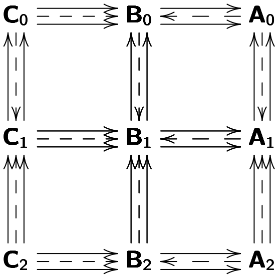
The squares “sequentially commute”

3×3 Diagram of Categories

We have cells χ, δ, μ, τ

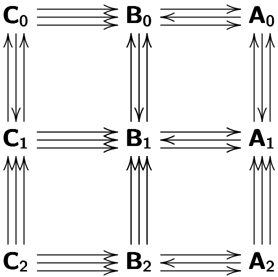


3 × 3 Diagram of Categories



The mixed (dashed and solid) squares sequentially commute

Intercategory



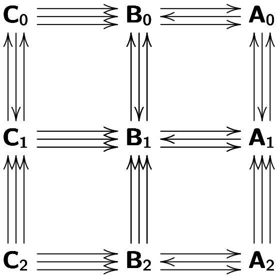
(1) Each column has the structure of a weak double category, $\mathbb{A}, \mathbb{B}, \mathbb{C}$
(so $\mathbf{A}_2 = \mathbf{A}_1 \times_{A_0} \mathbf{A}_1$, etc., $\mathfrak{a}', \mathfrak{l}', \mathfrak{r}'$)

commutativities $\Rightarrow$ $\mathbb{C} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathbb{B} \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \mathbb{A}$ strict functors

(2) τ and μ make id : $\mathbb{A} \longrightarrow \mathbb{B}$ a lax functor
δ and χ make $m : \mathbb{C} \longrightarrow \mathbb{D}$ a lax functor

(3) $\mathbb{C} \rightrightarrows \mathbb{B} \rightleftarrows \mathbb{A}$ is a pseudocategory in $\mathcal{D}bl\mathcal{L}ax$
(so $\mathbb{C} = \mathbb{B} \times_{\mathbb{A}} \mathbb{D}$, $\mathfrak{a}, \mathfrak{l}, \mathfrak{r}$)

Intercategory (equiv.)



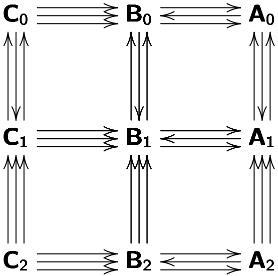
(1) Each row has the structure of a weak double category $\mathbb{X}_0, \mathbb{X}_1, \mathbb{X}_2$
(so $\mathbf{C}_i = \mathbf{B}_i \times_{\mathbf{A}_i} \mathbf{B}_i$, $\mathfrak{a}, \mathfrak{l}, \mathfrak{r}$)

commutativities $\Rightarrow \mathbb{X}_2 \overset{P_1}{\underset{P_2}{\rightrightarrows}} \mathbb{X}_1 \overset{D_0}{\underset{D_1}{\rightrightarrows}} \mathbb{X}_0$ strict functors

(2) τ and δ make $\mathrm{Id} : \mathbb{X}_0 \longrightarrow \mathbb{X}_1$ a colax functor
μ and χ make $M : \mathbb{X}_2 \longrightarrow \mathbb{X}_1$ a colax functor

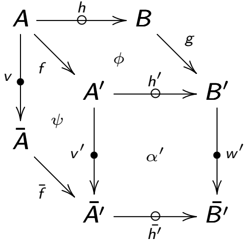
(3) $\mathbb{X}_2 \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; \mathbb{X}_1 \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \mathbb{X}_0$ is a pseudocategory in $\mathcal{D}bl\mathcal{C}olax$
(so $\mathbb{X}_2 = \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1$ and $\mathfrak{a}', \mathfrak{l}', \mathfrak{r}'$)

Geometric Representation



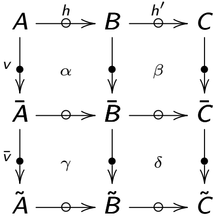
Intercategory $\mathfrak{A}$ has

- Objects = objects of $\mathbf{A}_0$
- Transversal arrows = morphisms of $\mathbf{A}_0$
- Horizontal arrows = objects of $\mathbf{B}_0$
- Vertical arrows = objects of $\mathbf{A}_1$
- Horizontal cells = morphisms of $\mathbf{B}_0$
- Lateral cells = morphisms of $\mathbf{A}_1$
- Basic cells = objects of $\mathbf{B}_1$
- Cubes = morphisms of $\mathbf{B}_1$



Composition

- Can be composed in all three directions
- Transversal is strictly associative and unitary $(\cdot, 1)$
- Horizontal (vertical) is associative and unitary up to coherent transversal isomorphism ($\circ$, id, resp. $\bullet$, Id)
- Horizontal and lateral cells compose in two directions and satisfy interchange
- Basic cells compose horizontally and vertically and have lax interchange



$$\chi : (\alpha \circ \beta) \bullet (\gamma \circ \delta) \longrightarrow (\alpha \bullet \gamma) \circ (\beta \bullet \delta)$$

$$\chi : \frac{\alpha|\beta}{\gamma|\delta} \longrightarrow \left.\frac{\alpha}{\gamma}\right| \frac{\beta}{\delta}$$

- Degenerate interchangers

$$\mu : \frac{\mathrm{id}_v}{\mathrm{id}_{\bar{v}}} \longrightarrow \mathrm{id}_{v \bullet \bar{v}} \qquad \delta : \mathrm{Id}_{h \circ h'} \longrightarrow \mathrm{Id}_h \,|\, \mathrm{Id}_{h'} \qquad \tau : \mathrm{Id}_{\mathrm{id}_A} \longrightarrow \mathrm{id}_{\mathrm{Id}_A}$$

Morphisms

A morphism of intercategories $F : \mathfrak{A} \longrightarrow \mathfrak{B}$ takes all the elements of $\mathfrak{A}$ to similar ones of $\mathfrak{B}$ and preserves domains and codomains

- Transversal composition is strictly preserved
- It can be
  - colax on the horizontal and lax on the vertical
  - colax on both horizontal and vertical
  - lax on both horizontal and vertical
- The lax-lax with colax-lax form a strict double category $\mathbb{I}\,\mathbb{C}\mathrm{at}$
- The colax-lax with colax-colax form a strict double category $\mathbb{I}\,\mathbb{C}\mathrm{at}^*$

Theorem

There is a strict triple category $\mathfrak{ICat}$ whose objects are intercategories, transversal arrows are colax-lax functors, horizontal arrows lax-lax functors, vertical arrows colax-colax functors, two dimensional cells as for $\mathbb{D}\mathrm{bl}$, and commutative cubes as 3-cells

Duoidal Categories

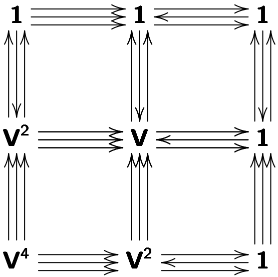
Aguiar & Mahajan $\longrightarrow$ 2-monoidal categories (Book, Ch. 6)
Booker & Street $\longrightarrow$ Duoidal (Tannaka Duality... TAC)
Böhm, Chen, Zhang $\longrightarrow$ (Hopf Monoids in Duoidal Categories, arXiv 2012)
$(\mathbf{V}, \otimes, \boxtimes, I, J)$

- $\otimes$ is lax for $\boxtimes$ (also I)
- $\boxtimes$ is colax for $\otimes$ (also J)
- Pseudomonoid in $\mathcal{M}\mathrm{on}_{\mathsf{lax}}$
- Pseudomonoid in $\mathcal{M}\mathrm{on}_{\mathsf{colax}}$

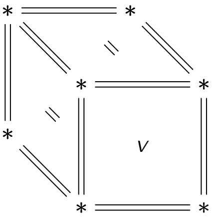
- $\chi : (A \otimes B) \boxtimes (C \otimes D) \longrightarrow (A \boxtimes C) \otimes (B \boxtimes D)$
- $\delta : J \longrightarrow J \otimes J$
- $\mu : I \boxtimes I \longrightarrow I$
- $\tau : J \longrightarrow I$

Proposition

A duoidal category is "the same as" an intercategory with only one object, only identity transversal, horizontal, vertical arrows, horizontal and lateral cells



- Lax-lax, colax-lax and colax-colax morphisms are called double lax, bilax and double colax by Aguiar & Mahajan
- A general cube looks like (with $w \longrightarrow v$ inside)

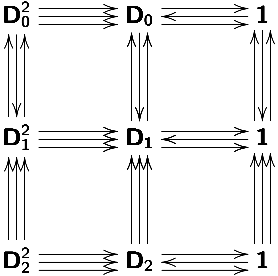


Monoidal Double Categories

- Shulman in "Constructing Symmetric Monoidal Bicategories" arXiv (2010) introduces *monoidal double categories*
- They are pseudomonoids in the 2-category of weak double categories and strong morphisms
- Can be considered as pseudocategories

$$\mathbb{D}^2 \rightrightarrows\!\!\!\rightarrow \mathbb{D} \rightleftarrows\!\!\!\rightarrow \mathbf{1}$$

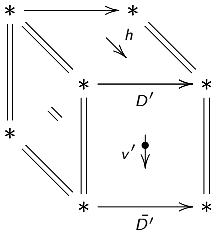
  in either $\mathcal{D}bl\mathcal{L}ax$ or $\mathcal{D}bl\mathcal{C}olax$
- $(\mathcal{D}bl\mathcal{L}ax)$



Proposition

A monoidal double category is "the same as" an intercategory with one object and only identity transversal and vertical (horizontal) arrows and lateral (resp. horizontal) cells. Furthermore the interchangers χ, δ, μ, τ are isomorphisms

- A general cube looks like (with double cell inside)



- There are good examples when the χ is not an isomorphism, e.g. a double category with a lax choice of products

Locally Cubical Bicategories

Garner & Gurski "The low-dimensional structures formed by tricategories" Math. Proc. Camb. Phil. Soc. (2009)

> *Like a tricategory, it has* 0-, 1-, 2- *and* 3-*cells; but the* 2-*cells come in two different kinds, vertical and horizontal, whilst the* 3-*cells are cubical in nature. Moreover, the coherence axioms that are to be satisfied are of a bicategorical, rather than a tricategorical kind, and so the resultant structure is computationally more tractable than a tricategory.*

A *locally cubical bicategory* $\mathfrak{B}$ is a category weakly enriched in the monoidal 2-category of weak double categories with strong morphisms and horizontal transformations

- For each pair of objects we have a weak double category $\mathfrak{B}(A, B)$
- $\otimes : \mathfrak{B}(A, B) \times \mathfrak{B}(B, C) \longrightarrow \mathfrak{B}(A, C)$
- $I_A : \mathbf{1} \longrightarrow \mathfrak{B}(A, A)$
- Associative and unitary up to coherent isomorphism

We get a pseudocategory object in $\mathcal{D}bl\mathcal{S}t$

$$\sum_{A,B,C} \mathfrak{B}(A,B) \times \mathfrak{B}(B,C) \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \sum_{A,B} \mathfrak{B}(A,B) \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathrm{Ob}(\mathfrak{B})$$

So it is an intercategory that looks like

$$\begin{array}{ccccc}
\mathbf{C}_0 & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} & \mathbf{B}_0 & \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} & \mathrm{Ob}(\mathfrak{B}) \\
\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow \\
\mathbf{C}_1 & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} & \mathbf{B}_1 & \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} & \mathrm{Ob}(\mathfrak{B}) \\
\uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow \\
\mathbf{C}_2 & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} & \mathbf{B}_2 & \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} & \mathrm{Ob}(\mathfrak{B})
\end{array}$$

i.e. one in which the horizontal and vertical arrows as well as the lateral cells are identities. Furthermore the interchangers χ, δ, μ, τ are isomorphisms

Verity Double Bicategories

- Solution to the problem of defining double categories that are weak in both horizontal and vertical directions
- Formalize special cells in horizontal and vertical bicategories, $\mathcal{H}$ and $\mathcal{V}$, with the same objects
- For every

  $$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} \\ C & \xrightarrow[h']{} & D \end{array}$$

  give a set of "squares", taken together give a discrete bifibration

  $$\mathbf{V} \times \mathbf{H} \xleftarrow{(\partial_1,\partial_1)} \mathbf{S} \xrightarrow{(\partial_0,\partial_0)} \mathbf{V} \times \mathbf{H}$$

- Double bicategories can be identified with intercategories with identity transversal arrows and satisfying the discrete bifibration property
- Verity – Thesis TAC reprints
- Morton – Extended TQFT's and Quantum Gravity

True Gray Categories

- Gordon, Power, Street, *Coherence for Tricategories*, Memoirs AMS
- A Gray category is a tricategory in which everything is strict except interchange, which is up to coherent isomorphism
- It is a category enriched in **Gray** the category of 2-categories and 2-functors with the *Gray* tensor product $\mathcal{A} \otimes \mathcal{B}$ which classifies

$$F : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$$

  - 2-functors in each variable separately
  - coherent isomorphisms

$$\begin{array}{ccc}
F(A,B) & \xrightarrow{F(f,B)} & F(A',B) \\
{\scriptstyle F(A,g)}\downarrow & \overset{h(f,g)}{\Longrightarrow} & \downarrow{\scriptstyle F(A',g)} \\
F(A,B') & \xrightarrow[F(f,B')]{} & F(A',B')
\end{array}$$

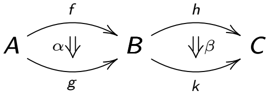
- Gray's original definition didn't have isomorphisms [Gray, Formal Category Theory, SLN 391]
  This gives another (non symmetric) $\otimes$ on 2-categories
  Call enriched categories relative to this $\otimes$, *true Gray categories*
  Thus a true Gray category has homs which are 2-categories $\mathcal{A}(A, B)$ and a 2-functor

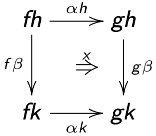
$$\mathcal{A}(A,B) \otimes \mathcal{A}(B,C) \longrightarrow \mathcal{A}(A,C)$$

  giving composition, i.e. a "Gray 2-functor of two variables"

$$\mathcal{A}(A,B) \times \mathcal{A}(B,C) \longrightarrow \mathcal{A}(A,C)$$

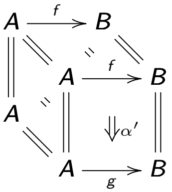
- This means that a true Gray category has objects, arrows, 2-cells and 3-cells with domains and codomains like for 3-categories
- The 2-cells and 3-cells compose strictly within their hom 2-categories
- The arrows also compose strictly, but there is no "horizontal" composition of 2-cells across the homs, just whiskering
- Interchange doesn't hold – there is only a comparison





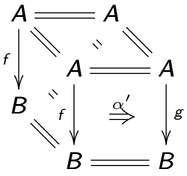
Either choice gives a strictly associative and unitary composition of 2- and 3-cells. The top choice is lax and the bottom is colax
In fact we get three different ways of making a true Gray category into an intercategory

$\mathfrak{A}_l$



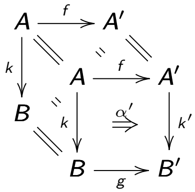
$(\alpha \longrightarrow \alpha')$

$\mathfrak{A}_c$



$(\alpha \longrightarrow \alpha')$

$\mathfrak{A}_s$



$(\alpha \longrightarrow \alpha')$