The Most General Distributive Law in the World

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DISTRIBUTIVE LAW FOR RINGS

$$a(b+c) = ab + ac \quad , \quad (b+c)a = ba + ca$$

$$(a_1 + a_2 + \cdots a_k)(b_1 + b_2 + \cdots b_l)(c_1 + c_2 + \cdots c_m)$$

$$= a_1 b_1 c_1 + a_1 b_1 c_2 + \cdots \qquad \text{(ALL CHOICES)}$$

$$= \sum_{r,s,t} a_r b_s c_t$$

I FINITE SET, $\forall i \in I$ FINITE SET J_i

$\sum J_i = J \xrightarrow{p} I$, $\langle a_j \rangle_{j \in J}$ ELTS OF R

$$\prod_{i \in I} \sum_{j \in J_i} a_j = \sum_{I \xrightarrow{\sigma} J} \prod_{i \in I} a_{\sigma(i)}$$

$$\begin{tikzcd} I \arrow[r, "\sigma"] \arrow[dr, equal] & J \arrow[d, "p"] \\ & I \end{tikzcd}$$

$$\begin{tikzcd} & R^J \arrow[r, "\Sigma_p"] \arrow[dl, "ev^*"'] & R^I \arrow[dd, "\Pi_I"] \\ R^{I \times \Pi J_i} \arrow[d, "\Pi_I"'] & & \\ R^{\Pi J_i} \arrow[rr, "\Sigma_{\Pi J_i}"'] & & R \end{tikzcd}$$

$$I \times \prod_i J_i \xrightarrow{ev} J$$

EVALUATION

SUBOBJECT LATTICE IN A TOPOS

$p : J \to I$ MORPHISM IN TOPOS $\underline{E}$

$\langle A_j \rangle$ FAMILY OF SUBOBJECTS OF A IN $\underline{E}$

FOR $a \in A$

$$a \in \bigcap_{i \in I} \bigcup_{p(j)=i} A_j \iff \forall i \left(a \in \bigcup_{p(j)=i} A_j \right)$$

$$\iff \forall i\, \exists j \left(p(j) = i \ \&\ a \in A_j \right)$$

NOT UNIQUE & NO CAN CHOICE

DON'T GET A FUNCTION $I \to J$

GET A LOCAL SECTION

$$\begin{tikzcd} & J \arrow[d, "p"] \\ K \arrow[ur, "\sigma"] \arrow[r, two heads] & I \end{tikzcd}$$

$$\bigcap_{i \in I} \bigcup_{p(j)=i} A_j = \bigcup_{\substack{K \xrightarrow{\sigma} J \\ K \twoheadrightarrow I}} \bigcap_{k \in K} A_{\sigma(k)}$$

with $p\sigma$ equal to $K \twoheadrightarrow I$,

$K \subseteq I \times J$

LINEAR ALGEBRA	COCOMPLETE CATEGORIES
VECT SP $+$ $\cdot$	COCOMP. CAT $\varinjlim$, $I \cdot A$
$\mathbb{R}$	$\underline{Set}$
$\mathbb{R}^n$	$\underline{Set}^{\mathbb{C}^{op}}$
LINEAR TRANSF.	COCONTINUOUS FUNCTOR
$\otimes$	$\otimes$ CLASSIFIES BI-CTS
DUAL	$\underline{A}^* = \underline{Cts}(\underline{A}, \underline{Set})$
HOM	$\underline{Cts}(\underline{A}, \underline{B})$
FIELD	TOPOS
MATRIX	PROFUNCTOR $P : \mathbb{C}^{op} \times \mathbb{D} \to \underline{Set}$
TRACE	$\int^{c} P(c, c)$
DISTRIBUTIONS	$\underline{A} \to \underline{Set}$ CTS
SYMMETRIC ALG	$\Sigma \underline{A}$ CLASSIFIES DIST.
FINITE SET	SMALL CAT
	$(\underline{Set}^{\mathbb{C}^{op}})^* \simeq \underline{Set}^{\mathbb{C}}$ (= REFLEXIVE)

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FIELD $\overset{?}{\longleftrightarrow}$ TOPOS

ALGEBRA

COMM. MONOID IN $\underline{\text{VECT}}$

$$\frac{\underline{A} \otimes \underline{A} \xrightarrow{\mu} \underline{A}}{\underline{A} \times \underline{A} \xrightarrow{M} \underline{A}}$$

COCTS IN EACH VAR

$\approx$ MONOIDAL CLOSED CAT

Q: HOW DO WE GET M RT. ADJOINT TO Δ?

EVEN WITH THAT, ONLY GET CART. CLOSED.

Q: WHAT ABOUT OTHER EXACTNESS PROPERTIES? E.G. COPRODUCTS UNIV.

Q: WHAT ARE EXACTNESS PROPERTIES OF PB?

$V^{\otimes m}$ CLASSIFIES MULI LIN MAPS

$$\frac{V^{\otimes m} \longrightarrow W}{f : V^m \longrightarrow W}$$

$$f\left(\left\langle \sum_{p(j)=i} a_j \right\rangle_j\right) = \sum_{\substack{I \xrightarrow{\sigma} J \\ \searrow\!\!\searrow \ \downarrow p \\ I}} f(\langle a_{\sigma(i)} \rangle)$$

$$\frac{\underline{A}^{\otimes \mathbb{C}} \longrightarrow \underline{B}}{\underline{A}^{\mathbb{C}} \longrightarrow \underline{B}}$$

s.t. ?

THE $\mathbb{C}$ ARE LIKE FIN SETS

EXAMPLES

1. $$\varinjlim_i B_i \times_A \varinjlim_j C_j \cong \varinjlim_{i,j} (B_i \times_A C_j)$$

2. $$\prod_{i\in I} \sum_{p(j)=i} A_j \cong \sum_{\substack{I \xrightarrow{\sigma} J \\ \| \quad \downarrow p \\ I}} \prod_{i\in I} A_{\sigma(i)}$$

3. $$\varprojlim_i \varinjlim_j A_{ij} \cong \varinjlim_j \varprojlim_i A_{ij}$$
(FINITE) (FILTERED)

↖ THESE HOLD IN ANY TOPOS

4. $\Delta_i : \mathbb{J}_i \longrightarrow \underline{\mathrm{SET}}$, $i \in I$ (INFINITE)

$$\prod_{i\in I} \varinjlim \Delta_i \not\cong \varinjlim_{\langle J_i\rangle \in \prod_I \mathbb{J}_i} \prod_{i\in I} \Delta_i J_i$$

$\rotatebox[origin=c]{90}{$\in$}$

$$\langle [x_i \in \Delta_i J_i] \rangle_{i\in I} \longleftarrow\!\shortmid [\langle x_i \in \Delta_i J_i \rangle_{i\in I}]$$

ONTO

NOT 1-1

THE SET-UP

$\mathbb{I}$ SMALL CAT $\qquad \mathbb{I} \longrightarrow \mathrm{CAT}$ (PSEUDO) FUNCT

$$\begin{tikzcd}
I \arrow[r, mapsto] \arrow[d, "i"'] & \mathbb{J}_I \arrow[d, "i_*"] \\
I' \arrow[r, mapsto] & \mathbb{J}_{I'}
\end{tikzcd}$$

GIVES A COFIBRATION

$$\begin{tikzcd}
\mathbb{J} \arrow[d, "p"] \\
\mathbb{I}
\end{tikzcd}
\qquad (I, J \in \mathbb{J}_I) \xrightarrow{(i,j)} (I', J' \in \mathbb{J}_{I'})$$

$$I \xrightarrow{i} I'$$

$$i_* J \xrightarrow{j} J'$$

TAKE A DIAGRAM $\Phi : \mathbb{J} \longrightarrow \mathrm{SET}$

EQUIVALENTLY

$$\begin{tikzcd}
\mathbb{J}_I \arrow[dr, "\Phi_I"] \arrow[dd, "i_*"'] \arrow[dd, phantom, shift left=4ex, "\varphi_i \Downarrow"] & \\
 & \mathrm{SET} \\
\mathbb{J}_{I'} \arrow[ur, "\Phi_{I'}"'] &
\end{tikzcd}$$

WHICH INDUCES

$$\begin{tikzcd}
\varinjlim \Phi_I \arrow[d] \\
\varinjlim \Phi_{I'}
\end{tikzcd}$$

WANT TO WRITE

$$\varprojlim_{I} \varinjlim_{\mathbb{J}_I} \Phi_I \cong \varinjlim_{?} \varprojlim_{?} \; ?$$

DEF: A FUNCTOR $Q : \tilde{\mathbb{I}} \to \mathbb{I}$ IS CALLED A **CONNECTED COVER** IF

(1) EACH FIBRE $Q^{-1}(I)$ IS CONNECTED AND NONEMPTY

(2) ONTO ON ARROWS

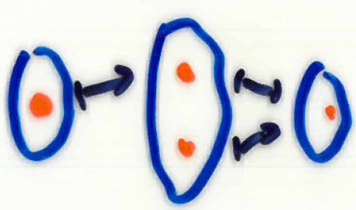
PROP: $Q : \tilde{\mathbb{I}} \to \mathbb{I}$ IS A CONNECTED COVER IFF FOR EVERY $F : \mathbb{C} \to \mathbb{I}$ WITH $\mathbb{C}$ CONNECTED, THE PULLBACK $F^* \tilde{\mathbb{I}}$ IS CONNECTED.

COR: CONNECTED COVERS ARE:

(1) CLOSED UNDER COMPOSITION,

(2) STABLE UNDER PULLBACK.

Ex: NO CANCELLATION PROPERTIES



DEF: GIVEN A COFIBRATON $P : \mathbb{J} \to \mathbb{I}$, a **LOCAL SECTION OF P** IS A CONNECTED COVER $Q : \tilde{\mathbb{I}} \to \mathbb{I}$ AN FUNCTOR $\Gamma : \tilde{\mathbb{I}} \to \mathbb{J}$ SUCH THAT

$$\begin{tikzcd} & \mathbb{J} \arrow[d, "P"] \\ \tilde{\mathbb{I}} \arrow[ur, "\Gamma"] \arrow[r, "Q"'] & \mathbb{I} \end{tikzcd}.$$

A **MORPHISM OF LOCAL SECTIONS** IS

$$\begin{tikzcd} \tilde{\mathbb{I}} \arrow[dr, "\Gamma"] \arrow[d, "F"'] & \\ \tilde{\mathbb{I}}' \arrow[r, "\Gamma'"'] \arrow[u, phantom, shift left=4, "\varphi \Uparrow"] & \mathbb{J} \end{tikzcd}$$

SUCH THAT

(1) $Q'F = Q$

(2) $P\varphi = \mathrm{id}_Q$

THM: LET $P : \mathbb{J} \to \mathbb{I}$ BE A COFIBRATION AND $\Phi : \mathbb{J} \to \mathrm{SET}$ A DIAGRAM. THEN

$$\boxed{\varprojlim_{I} \varinjlim_{\mathbb{J}_I} \Phi_I \cong \varinjlim_{\substack{\begin{tikzcd}[ampersand replacement=\&] \& \mathbb{J} \arrow[d, "P"] \\ \tilde{\mathbb{I}} \arrow[ur, "\Gamma"] \arrow[r, "Q"'] \& \mathbb{I} \end{tikzcd} \\ \#\tilde{\mathbb{I}} = \#\mathbb{I}}} \varprojlim_{\tilde{\mathbb{I}}} \Phi\Gamma}$$

"PROOF" $\varinjlim_{\Gamma/\mathbb{J}} \varprojlim_{\tilde{\mathbb{I}}} \Phi\Gamma \xrightarrow{\ \Theta\ } \varprojlim_{I} \varinjlim_{\mathbb{J}_I} \Phi$

$$\begin{tikzcd} & \mathbb{J} \arrow[d, "P"] \\ \tilde{\mathbb{I}} \arrow[ur, "\Gamma"] \arrow[r, "Q"'] & \mathbb{I} \end{tikzcd}$$

AN ELEMENT ON LHS IS AN EQUIVALENCE CLASS OF PAIRS

$$\begin{tikzcd} & \mathbb{J} \arrow[d, "P"] \\ \tilde{\mathbb{I}} \arrow[ur, "\Gamma"] \arrow[r, "Q"'] & \mathbb{I} \end{tikzcd} , \quad \langle x_{\tilde{I}} \in \Phi\Gamma\tilde{I} \rangle$$

1. $\forall I \in \mathbb{I}$ PICK $\tilde{I}$ S.T. $Q\tilde{I} = I$

THEN $\Gamma\tilde{I} \in \mathbb{J}_I$ SO TAKE $[x_{\tilde{I}} \in \Phi\Gamma\tilde{I}] \in \varinjlim_{\mathbb{J}_I} \Phi_I$

2. IF $Q\tilde{I}' = I$ TOO, THEN $\exists$ PATH

$$\tilde{I} \leftarrow \tilde{I}_1 \rightarrow \cdots \quad \tilde{I}_n \rightarrow \tilde{I}' \quad \text{IN } \mathbb{J}_I$$

$$\begin{tikzcd} \Phi\Gamma\tilde{I} & \Phi\Gamma\tilde{I}_1 \arrow[l] \arrow[r] & \cdots & \Phi\Gamma\tilde{I}_n \arrow[r] & \Phi\Gamma\tilde{I}' \\ x_{\tilde{I}} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & x_{\tilde{I}_1} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[l, mapsto] \arrow[r, mapsto] & \cdots & x_{\tilde{I}_n} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & x_{\tilde{I}'} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

SO $[x_{\tilde{I}}] = [x_{\tilde{I}'}]$ IN $\varinjlim_{\mathbb{J}_I} \Phi_I$

TAKE THIS TO BE $[y_I]$

IT IS EASY TO CHECK THAT

3. $[y_I]$ IS INDEPENDENT OF THE REPRESENTATIVE OF CLASS ON LEFT

4. THE FAMILY $\langle [y_I] \rangle$ IS COMPATIBLE IN I & THUS $\in \varprojlim_{I} \varinjlim_{\mathbb{J}_I} \Phi_I$

THIS GIVES $\Theta\left[Q, \Gamma, \langle x_I \rangle\right] = \langle [y_I] \rangle$

$$\Theta : \varinjlim_{\substack{\Gamma \nearrow \mathbb{J} \\ \tilde{\mathbb{I}} \xrightarrow{Q} \mathbb{I}}} \varprojlim_{\tilde{\mathbb{I}}} \Phi\Gamma \longrightarrow \varprojlim_{I} \varinjlim_{\mathbb{J}_I} \Phi_I$$

Θ IS ONTO

LET $\left\langle [y_I \in \Phi J_I] \right\rangle_{I \in \mathbb{I}} \in \varprojlim_{I} \varinjlim_{J_I} \Phi_I$

PICK A REPRESENTATVE $y_I \in \Phi J_I$ FOR EA. CL.

FOR EVERY $i : I \longrightarrow I'$, $\tilde{i} : J_I \longrightarrow i_* J_I$

$$\Phi(\tilde{i})(y_I) \in \Phi i_* J_I \sim y_{I'} \in \Phi J_{I'}$$

PICK A PATH

$$i_* J_I \longrightarrow J_{i,1} \leftarrow J_{i,2} \rightarrow \cdots \ J_{i,n_i} \leftarrow J_{I'}$$

$$\Phi(\tilde{i})(y_I) \mapsto y_{i,1} \mapsfrom y_{i,2} \mapsto \cdots \ y_{i,n_i} \mapsfrom y_{I'} \quad \text{ELTS OF } \Phi$$

CONSTRUCT $Q : \tilde{\mathbb{I}} \longrightarrow \mathbb{I}$

OBJECTS

ONE $\tilde{I}$ $\forall I \in \mathbb{I}$

ONE $\widetilde{(i,r)}$ $\forall J_{i,r}$

ARROWS

$\tilde{I} \rightarrow \widetilde{(i,1)}$, $\tilde{I}' \longrightarrow \widetilde{(i,n_i)}$

$\widetilde{(i,r)} \rightarrow \widetilde{(i,r \pm 1)}$ AS $\nearrow$

NO COMPOSITS

$Q(\tilde{I}) = I$

$Q(\widetilde{(i,r)}) = I'$

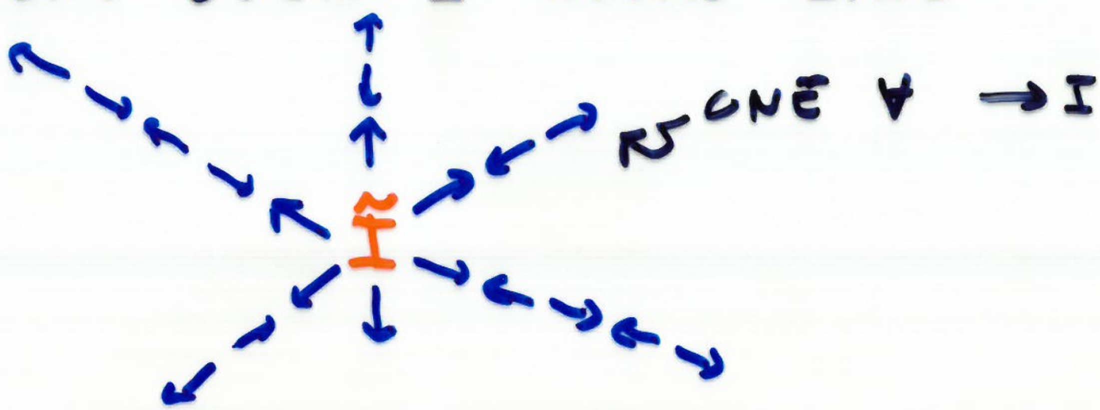
$Q(\tilde{I} \rightarrow \widetilde{(i,1)}) = i$

Q(OTHERS) = IDENTITIES

$Q : \tilde{\mathbb{I}} \longrightarrow \mathbb{I}$ IS A CONNECTED COVER AND

$\# \tilde{\mathbb{I}} = \# \mathbb{I}$

THE FIBRE OVER I LOOKS LIKE



DEFINE $\Gamma : \tilde{\mathbb{I}} \longrightarrow \mathbb{J}$ BY

$\Gamma(\tilde{I}) = J_I \qquad \Gamma(\widetilde{i,r}) = J_{i,r}$

Γ(ARROWS) = THOSE IN CHOSEN PATHS

THE ELEMENTS $y_I \in \Phi J_I$, $y_{i,r} \in \Phi J_{i,r}$

GIVE AN ELEMENT $\langle x \rangle \in \varprojlim_{\tilde{\mathbb{I}}} \Phi \Gamma$

& $\theta[Q, \Gamma, \langle x \rangle] = \langle [y_I] \rangle$

θ IS ONE-ONE

$$\begin{tikzcd} & \mathbb{J} \arrow[d, "P"] \\ \tilde{\mathbb{I}}_i \arrow[ur, "\Gamma_i"] \arrow[r, "Q_i"'] & \mathbb{I} \end{tikzcd} \qquad i=1,2$$

$\langle x_{\tilde{I}_1} \in \Phi\Gamma_1\tilde{I}_1 \rangle$

$\langle z_{\tilde{I}_2} \in \Phi\Gamma_2\tilde{I}_2 \rangle$

WHICH θ IDENTIFIES IN $\underset{\overleftarrow{I}}{\text{Lim}}\ \underset{\overrightarrow{\mathbb{J}_S}}{\text{Lim}}\ \Phi_I$

$\forall I$ PICK $\tilde{I}_1, \tilde{I}_2$ ST $Q_i\tilde{I}_i = I$

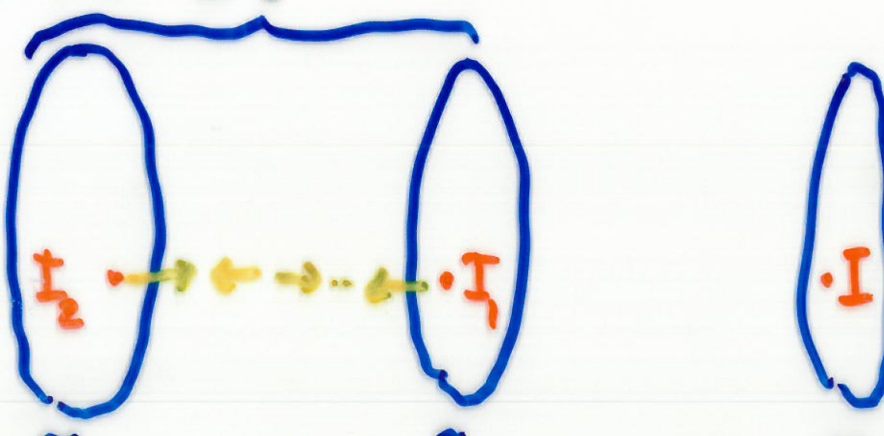
THEN $x_{\tilde{I}_1} \in \Phi\Gamma_1\tilde{I}_1 \sim z_{\tilde{I}_2} \in \Phi\Gamma_2\tilde{I}_2$

PICK A PATH

$$\Gamma_1\tilde{I}_1 \to J_1 \leftarrow J_2 \to \cdots\ J_m \leftarrow \Gamma_2 \tilde{I}_2$$

$$x_{\tilde{I}_1} \mapsto y_1 \mapsfrom y_2 \mapsto \cdots\ y_m \mapsfrom z_{\tilde{I}_2} \quad \text{IN } \Phi$$

BUILD $\tilde{\mathbb{I}}_J$



$\tilde{\mathbb{I}}_1$ $\quad$ $\tilde{\mathbb{I}}_2$ $\quad$ $\mathbb{I}$

CONSTRUCT

$\Gamma : \mathbb{I} \to \mathbb{J}$

OUT OF J_i

AND FAM OF

ELTS FROM x, y, z

RESTRICT to x, y

SPECIAL CASES

1. P IS $P_1 : \mathbb{I} \times \mathbb{K} \longrightarrow \mathbb{I}$

A LOCAL SECTION IS JUST $\tilde{\mathbb{I}} \xrightarrow{\Gamma} \mathbb{K}$

$$\varprojlim_{I} \varinjlim_{K} \Phi(I,K) \cong \varinjlim_{\substack{\tilde{\mathbb{I}} \xrightarrow{\Gamma} \mathbb{K} \\ \tilde{\mathbb{I}} \xrightarrow{Q} \mathbb{I}}} \varprojlim_{\tilde{\mathbb{I}}} \Phi(Q\hat{I}, \Gamma\hat{I})$$

2. $P : \mathbb{J} \longrightarrow \mathbb{I}$ IS A **DISCRETE** COFIBRATION

LOCAL SECTIONS ARE CONSTANT ON FIBRES AND CAN BE REPLACED BY ACTUAL SECTIONS (DISCRETE)

$$\varprojlim_{I} \sum_{J \in \mathbb{J}_I} \Phi(J) \cong \sum_{\langle J_I \rangle \in \varprojlim \mathbb{J}_I} \varprojlim_{I} \Phi(J_I)$$

FUNCTORIAL VERSION

$\mathbb{LS}(P)$ = CAT OF LOCAL SECTIONS OF P

$\mathbb{LS}(P) \longrightarrow \mathrm{CAT}$ GIVES COFIB

$$\begin{tikzcd} & \mathbb{J} \\ \tilde{\mathbb{I}} \arrow[ur] \arrow[dr, two heads] & & \mapsto & \tilde{\mathbb{I}} \\ & \mathbb{I} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}\mathrm{ov} \arrow[d, "R"] \\ \mathbb{LS}(P) \end{tikzcd}$$

OBJ OF $\mathbb{C}\mathrm{ov}$ IS $\tilde{I} \in \tilde{\mathbb{I}}$ $\begin{tikzcd} & \mathbb{J} \\ {} \arrow[ur, "\Gamma"] \arrow[dr, two heads, "Q"'] & \\ & \mathbb{I} \end{tikzcd}$

$\mathrm{Ev} : \mathbb{C}\mathrm{ov} \longrightarrow \mathbb{J}$

$$\begin{tikzcd} & \mathrm{Set}^{\mathbb{J}} \arrow[rr, "\varinjlim_P"] \arrow[dl, "\mathrm{Ev}^*"'] & & \mathrm{Set}^{\mathbb{I}} \arrow[dd, "\varprojlim"] \\ \mathrm{Set}^{\mathbb{C}\mathrm{ov}} \arrow[dr, "\varprojlim_R"'] & & & \\ & \mathrm{Set}^{\mathbb{LS}(P)} \arrow[rr, "\varinjlim"'] & & \mathrm{Set} \end{tikzcd}$$

Kan Extensions

If $P : \mathbb{J} \longrightarrow \mathbb{I}$ is not a cofibration we can take left Kan extension

$$\mathrm{Lan}_P \Phi(I) = \varinjlim_{PJ \to I} \Phi J$$

This ammounts to replacing P by

$$\partial_1 : (P, \mathbb{I}) \longrightarrow \mathbb{I}$$

Local sections of ∂_1 are local "pre-sections" of P

$$\begin{tikzcd} & \mathbb{J} \arrow[d, "P"] \\ \tilde{\mathbb{I}} \arrow[ur, "\Gamma"] \arrow[r, "Q"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] & \mathbb{I} \end{tikzcd}$$

$$\varprojlim_{\mathbb{I}} \mathrm{Lan}_P \Phi \cong \varinjlim_{\begin{array}{c} \tilde{\mathbb{I}} \xrightarrow{\Gamma} \mathbb{J} \\ \Downarrow\gamma \quad \downarrow \\ \tilde{\mathbb{I}} \xrightarrow{Q} \mathbb{I} \end{array}} \varprojlim_{\tilde{\mathbb{I}}} \Phi\Gamma$$