Mealy Morphisms
of
Enriched Categories

'Robert Pare'
Fredericton
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BACKGROUND

A MONOIDAL CATEGORY IS A CATEGORY \( \mathcal{V} \) EQUIPPED WITH A "TENOR PRODUCT" \( \otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \) AND A UNIT OBJECT \( I \) WHICH IS ASSOCIATIVE

\[ \alpha_{ABC} : A \otimes (B \otimes C) \overset{\sim}{\rightarrow} (A \otimes B) \otimes C \]

AND UNITARY

\[ \rho_A : A \otimes I \overset{\sim}{\rightarrow} A \quad \delta_A : I \otimes A \overset{\sim}{\rightarrow} A \]

THE \( \alpha, \delta, \rho \) SATISFYING SOME "COHERENCE CONDITIONS".

**Ex:** \( \mathcal{V} = \mathbb{Z} \), \( \otimes \) THE USUAL, \( I = \mathbb{Z} \).

**Ex:** \( \mathcal{V} = \text{Set} \), \( \otimes \) = CARTESIAN PRODUCT, \( I = 1 \).

**Ex:** \( \mathcal{V} = \text{Cat} \), \( \otimes \) = , \( I = 1 \).
A **V**-category **A** consists of a class of objects **Ob**A, for each pair of objects A, B an object A(A,B) ∈ **V**, for each object A an "identity morphism" \( \text{Id}_A : I \rightarrow A(A,A) \) and a "composition morphism" for each A, B, C

\[
\circ_{A,B,C} : A(B,C) \otimes A(A,B) \rightarrow A(A,C)
\]

satisfying associativity and unit laws.

**Example:** \( V = \text{Ab} \) a V-category is an additive category. E.g. \( \text{R-Mod} \).

A ring \( R \) is an \( \text{Ab} \)-category with one object.

**Example:** A set-category is an ordinary category.
**Example:** A **category** is called a **2-category**. It has **objects**, **arrows** and **2-cells**.

![Diagram](image)

**Example:** CAT itself is a 2-category - the 2-cells are natural transformations.

**Bicategory** is a weak 2-category: composition of arrows is only associative and unitary up to coherent isomorphisms.

**Example:** OBJ - rings, arrows $R \rightarrow S$, $S$-$R$ bimodules, 2-cells linear functions.

Composition:

![Diagram](image)
Example: A monoidal category can be considered as a one object bicategory.

Morphisms of $V$-categories

A $V$-functor $F: A \to B$ is given by a function $F: ObA \to ObB$, and for each pair of objects $A, A' \in A$, a $V$-morphism $F: A(A, A') \to B(FA, FA')$ satisfying the commutativity of 2 diagram expressing that $F$ preserves composition and units, in a way internal to $V$.

Example: $V = \text{Set}$ we get ordinary functors.

Example: $V = \text{Ab}$ " " additive " "

Example: $V = \text{Cat}$ " " 2-functors."
**LAX MORPHISMS**

For 2-categories and bicategories we might want to relax the preservation of composition and identities

\[ F : \text{A} \rightarrow \text{B} \]

\( \text{OBJ} \rightarrow \text{OBJ}, \text{ARR} \rightarrow \text{ARR}, \text{2-CELLS} \rightarrow \text{2-CELLS} \)

Preserving all domains & codomains but with added structure

\[ \begin{array}{c}
\text{FA} \xrightarrow{\mu} \text{FA'^{\prime}} \\
\downarrow \text{\mu}_{\text{FA}} \\
\text{FA''} \\
\text{F(\text{\mu}'\text{f})}
\end{array} \]

And

\[ \begin{array}{c}
\text{FA} \xrightarrow{\mu} \text{FA} \\
\downarrow \text{\mu}_{\text{FA}} \\
\text{F(\text{\mu}_A)}
\end{array} \]

Satisfying associativity and unit equations.

If \( \mu, \eta \) are identities we get **strict morphisms**

If \( \mu, \eta \) are isos we get **pseudo morphisms**

If \( \mu, \eta \) are not isos we get **lax morphisms**
In "Bicategories I", Bénabou remarked that a $V$-category is the same as a class $\mathcal{OBA}$ and a lax morphism of bicategories $A : \mathcal{OBA} \to V$.

$\mathcal{OBA}$: objects are $A, A', A'' \in \mathcal{OBA}$
unique 1-cell $A \to A' \forall A, A'$
only identity 2-cells

$V$: one object $\ast$
1-cells objects of $V$
composition $\ast \xrightarrow{V} \ast \xrightarrow{V'} \ast = \ast \xrightarrow{V' \circ V} \ast$

$A : A \xrightarrow{\ast} \ast$ $(A, A') : A \to A' \quad \mapsto \quad A(A, A') \in V$

\[
\begin{align*}
(A, A'', A'') : A \to A' \quad &\mapsto \quad A(A, A') \in V \\
A \xrightarrow{A} A' \quad &\mapsto \quad A(A, A') \xrightarrow{\ast} A(A', A'') \\
(A, A'', A') \quad &\mapsto \quad A(A, A') \xrightarrow{\ast} A(A', A'') \quad A(A', A'') \xrightarrow{\ast} A(A, A'') \\
(A', A'') \circ A(A, A') &\mapsto A(A, A'')
\end{align*}
\]
Lax and oplax transformations of lax functors were defined in "Bicategories I" but \( \mathcal{V} \)-functors were not considered in this context.

A \( \mathcal{V} \)-functor \( A \to B \) is a pair \( (F, \varphi) \)

\[
\begin{array}{c}
\begin{array}{c}
\mathcal{O}_{A} \xrightarrow{F} \mathcal{O}_{B} \\
\mathcal{O}_{A} \\
A \\
\end{array}
\end{array}
\begin{array}{c}
\mathcal{O}_{B} \\
\mathcal{O}_{B} \\
B \\
\end{array}
\begin{array}{c}
\begin{array}{c}
\varphi \\
\varphi \\
\end{array}
\end{array}
\begin{array}{c}
A \\
B \\
\end{array}
\end{array}
\]

where \( F \) is a function on objects and \( \varphi \) is a lax transformation \( B \circ F \to A \)

which is the identity on objects (c.f. ICON).

\[
\begin{array}{c}
\begin{array}{c}
B(FA) \xrightarrow{\varphi A} A(A) \\
B(FA') \xrightarrow{\varphi A'} A(A') \\
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\begin{array}{c}
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\begin{array}{c}
B(FA, FA') \xleftarrow{\varphi(A, A')} A(A, A') \\
B(FA, FA') \\
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\]

I.e. \( \varphi(A, A') : A(A, A') \to B(FA, FA') \) + conditions.
**Definition** Let \( A \) and \( B \) be \( \mathcal{V} \)-categories. A **Mealy morphism** \( (F, \varphi) : A \rightarrow B \) consists of:

- **Object functions** \( F : \text{Ob} A \rightarrow \text{Ob} B \), \( \varphi : \text{Ob} A \rightarrow \text{Ob} B \);
- For each pair \( A, A' \in A \) a morphism
  \[ \varphi(A, A') : A(A, A') \otimes \varphi_A \rightarrow \varphi_A \otimes B(FA, FA') ; \]

satisfying

\[ \tag{Mm1} \]
\[ \Sigma \otimes \varphi_A \xrightarrow{id \otimes \varphi_A} A(A, A) \otimes \varphi_A \]
\[ \xrightarrow{\varphi(A, A)} \]
\[ \text{Can} \]
\[ \xrightarrow{\varphi(A \otimes \varepsilon)} \varphi_A \otimes B(FA, FA) \]

\[ \tag{Mm2} \]
\[ A(A', A'') \otimes A(A, A') \otimes \varphi_A \otimes \varphi_A \xrightarrow{\varphi \otimes \varphi_A} A(A''', A') \otimes \varphi_A \]
\[ \xrightarrow{\varphi(A', A'') \otimes \varphi(FA, FA')} \]
\[ A(A', A'') \otimes \varphi_A' \otimes B(FA, FA') \]
\[ \xrightarrow{\varphi(FA'') \otimes B(FA', FA')} \]
\[ \varphi_A' \otimes B(\widehat{FA}, FA') \rightarrow \varphi_A'' \otimes B(\widehat{FA}, FA'') \]
Mealy morphisms are easily composed.

Composition is not strictly associative as it uses \( \otimes \). Need 2-cells.

**Definition** Let \((F, \varphi), (G, \gamma) : A \to B\) be Mealy morphisms. A Mealy cell \(t : (F, \varphi) \to (G, \gamma)\) is given by morphisms

\[
\lambda_A : \varphi_A \to \gamma_A \otimes B(FA, CA)
\]

satisfying

\[
(M_c) \quad A((A, A')) \otimes \varphi_A \xrightarrow{\varphi(2AA')} \varphi' \otimes B(FA, FA')
\]

\[
\begin{array}{ccc}
A((A, A')) \otimes \varphi_A & \xrightarrow{} & \varphi' \otimes B(FA, FA') \\
\downarrow & & \downarrow \\
A((A, A')) \otimes \gamma_A \otimes B(FA, CA) & \xrightarrow{\gamma(2AA') \otimes B(FA, CA)} & \gamma' \otimes B(FA, CA') \otimes B(FA, FA') \\
\gamma(2AA') \otimes B(FA, CA) & \downarrow & \gamma' \otimes B(FA, CA') \\
\gamma' \otimes B(FA, FA') \otimes B(FA, CA) & \rightarrow & \gamma' \otimes B(FA, CA') \\
\end{array}
\]
GIVEN MEALY MORPHISMS \( (F, \eta)_A \rightarrow B \) AND \\
\( (H, \eta) : B \rightarrow C \) THE COMPOSITE IS \( (HF, \eta \circ \eta F) \) \\
WHERE \( (\eta \circ \eta F)_A = \eta A \circ \eta FA \) AND \( (\eta \circ \eta F)(A, A') \) IS \\
\[ A(A, A') \circ FA \circ \eta FA \rightarrow FA' \circ B(FA, FA') \circ \eta FA \]
\[ \eta(A, A') \circ 1 \]
\[ 1 \circ \eta(FA, FA') \]
\[ \eta FA' \circ \eta FA' \circ C(HFA, HFA') \].
Theorem \( V \)-categories, Mealy morphisms and Mealy cells form a bicategory \( V\text{-Mealy} \).

Example Let \( I \) be the \( V \)-category with one object \( * \) and \( I(*,*) = I \). For any \( V \)-category \( B \), a Mealy morphism \( I \to B \) is determined by a pair \( (V, B) \) \( V \) in \( V \), \( B \) in \( B \).

A Mealy cell \( b: (V, B) \to (V', B') \) is a morphism \( b: V \to V' \otimes_o B(B, B') \).

This describes the category \( \tilde{B}_0 := V\text{-Mealy}(I, B) \).

It is the underlying category of a \( V \)-category (assuming some closedness conditions on \( V \)) which will play an important role later.

Example \( V\text{-Mealy}(B, I) = V\text{-Cat}(B, V) \).
EXAMPLE

For \( V = \text{SET} \), the morphisms

\[ \varphi(A, A') : A(A, A') \times \varphi A \to \varphi A' \times B (FA, FA') \]

have two components:

\[ \varphi_1(A, A') : A(A, A') \times \varphi A \to \varphi A' \]

which make \( \varphi \) into a functor \( A \to \text{SET} \), and

\[ \varphi_2(A, A') : A(A, A') \times \varphi A \to B (FA, FA') \]

which give a lifting of \( F \) to a functor

\[ \text{El}(\Sigma) \]

In this way a Mealy morphism \( (F, \varphi) : A \to B \) can be viewed as a span in \( \text{CAT} \) where the left leg is a discrete opfibration and the right leg is object-wise constant on the fibres.

"PARTIAL FUNCTORS"
Example/Definition: A Mealy machine

[George Mealy 1955] has

1. A (finite) set $S$ of states
2. An input alphabet $\Sigma$
3. An output alphabet $\Lambda$
4. A transition function $t: \Sigma \times S \rightarrow S$
5. An output function $g: \Sigma \times S \rightarrow \Lambda$
6. A start state $s_0 \in S$

Run as follows:

Take a word in input alphabet $q_mq_{m-1}\ldots q_1$

Start in state $s_0$

New state $s_1 = t(q_1, s_0)$, output $b_1 = g(q_1, s_0)$

Then $s_2 = t(q_2, s_1)$, $b_2 = g(q_2, s_1)$

Etc.

So $q_mq_{m-1}\ldots q_1 \mapsto s_m\ldots s_2 b_1$

Extend $t$ and $g$ to free monoids

$\bar{t}: \Sigma^* \times S \rightarrow S$

or $\Sigma^* \times S \rightarrow S \times \Lambda^*$

$\bar{g}: \Sigma^* \times S \rightarrow \Lambda^*$
Example: A $V$-functor $F : A \to B$ corresponds exactly to a Mealy morphism $F_* : (F, I) : A \to B$, where $I$ represents the function $OBA \to OBY$ with constant value $I$. Composition is preserved $G(F) \cong (GF)$.

$V$-natural transformations $F \to K$ are in bijection with Mealy cells $F_* \to K$.

Thus we get a locally fully faithful embedding

$\gamma : V\text{-CAT} \to V\text{-Mealy}$
**Definition**  
**B** is left tensored if for every **V** in **V** and **B** in **B** there is an object \( V \otimes B \) in **B** and a morphism \( k : V \to B(B, V \otimes B) \) which mediates a bijection

\[
X \to B(V \otimes B, B')
\]

\[
X \otimes V \to B(B, B')
\]

For each \( X \) in **V** and \( B' \) in **B**.

**Theorem**  
Let \( A \) and **B** be **V**-categories with **B** tensored. Then the embedding

\[
() : V\text{-CAT}(A, B) \to V\text{-Mealy}(A, B)
\]

has a left adjoint.

**Proof**  
If \( (F, \eta) : A \to B \) is a Mealy morphism, then \( FA = \eta A \otimes FA \) is the object part of a **V**-functor which will be the reflection of \( (F, \eta) \). \( \square \)
Profunctors = The Categorician's Relations

A relation \( R : A \to B \) is a function \( A \times B \to \{T, F\} \).

The composite of \( A \to B \rightharpoonup C \) is \( (S \circ R) \iff \exists b (arb \land b \in C) \).

The identity \( I_A : A \to A \) is equality \( a \overset{I_A}{\leftrightarrow} a' \iff a = a' \).

This gives the category \( \text{Rel} \).

\( \text{Rel} \) is a 2-category: there is a 2-cell (unique) \( R \Rightarrow R' \) iff \( \forall a b \forall a' b' (arb \Rightarrow ar'b') \).

A function \( f : A \to B \) gives two relations

\( f_\#: A \to B \), \( a f_\# b \iff (fa = b) \)

\( f^\#: B \to A \), \( b f^\# a \iff (b = fa) \).
f^* is LEFT ADJOINT to f^•

For \( R : A \rightarrow B \) and \( S : B \rightarrow A \), R is LEFT ADJOINT to S if

\[ R \circ S \Rightarrow I_B \quad \text{and} \quad I_A \Rightarrow S \circ R \]

R is (comes from) a FUNCTION \( A \rightarrow B \) iff it has a RIGHT ADJOINT.

A PROFUNCTOR \( P : A \rightarrow B \) is a FUNCTOR \( P : B^{op} \times A \rightarrow \text{SET} \).

THE COMPOSITE of \( A \rightarrow B \xrightarrow{P} Q \rightarrow C \) is \( Q \circ P \)

is defined by

\[ Q \circ P(c, a) = \sum_B P(B, a) \times Q(c, B) \quad \text{(MATRIX MULTI)} \]

\[ = \int_B P(B, a) \times Q(c, B) \]

\[ \sum_{B, B'} P(B, a) \times B(B') \times Q(g_{B'}) \Rightarrow \sum_B P(B, a) \times Q(c, B) \Rightarrow Q \circ P(c, a) \quad \text{(MORPHISMS)} \]
The identity \( \text{Id}_A : A \rightarrow A \) is the hom functor \( A(-,-) : A^\text{op} \times A \rightarrow \text{Set} \).

A 2-cell \( P \Rightarrow P' : A \rightarrow B \) is a natural transformation \( \xi : P \Rightarrow P' : B^\text{op} \times A \rightarrow \text{Set} \).

We get a bicategory \( \text{Prof} \).

Notation: Write \( x : P(B,A) \) as \( B \xrightarrow{\eta} A \).

Functoriality says we can "compose" \( x \) with morphisms \( A \rightarrow A' \), \( \alpha x : B \rightarrow A' \) and morphisms \( B' \xrightarrow{\beta} B \), \( x \beta : B' \rightarrow A \). These composites are associative and unitary.

An element of \( \text{Qop}(C,A) \) is an equivalence class of pairs \( [C \xrightarrow{\eta} B \xrightarrow{\alpha} A] \). The equivalence is generated by

\[
\begin{align*}
&C \xrightarrow{\eta} B \xrightarrow{\alpha} A \\
&\downarrow \quad \downarrow \\
&C \xrightarrow{\eta'} B' \xrightarrow{\alpha'} A'
\end{align*}
\]
If we write the equivalence class as \( x \otimes y \), then \( x' \otimes y = x' \otimes b' y \).

Every functor \( F : A \to B \) determines two profunctors \( F^x : A \to B \), \( F^x (B, A) = B (B, FA) \) and \( F^y : B \to A \), \( F^y (A, B) = B (FA, B) \).

We have \( F^x \to F^y \).

Not quite true that an adjoint pair of profunctors \( P \to Q \) come from a functor.

B is Cauchy complete if this is true for every \( P : A \to B \) with a right adjoint.

For set based profunctors, B is Cauchy complete iff B has split idempotents.

The Cauchy completion of B is the category of profunctors \( 1 \to B \) with a right adjoint.
A relation \( R : A \to B \) can equally well be given by its graph \( MR \subseteq A \times B \).

Analogously, given a profunctor \( P : A \to B \) we construct its category of elements \( \text{El}(P) \).

An object of \( \text{El}(P) \) is \((B, A, x \in P(B, A))\).

A morphism \((B, A, x) \to (B', A', x')\) is a pair \(b : B \to B', \ q : A \to A'\) such that
\[ P(B, q)(x) = P(b, A') (x') \]

\[
\begin{array}{c}
B \xrightarrow{x} A \\
b \Downarrow \cong \Downarrow q \\
B' \xrightarrow{x'} A'
\end{array}
\]

We get \( \text{El}(P) \) \( A \to B \) a discrete bifibration

\[ P \cong \pi'_* \otimes \pi^* \]
MEALY MORPHISMS CAN BE VIEWED AS PROFUNCTORS

RECALL THAT A PROFUNCTOR \( P : A \rightarrow B \) IS GIVEN BY AN OBJECT FUNCTION \( P : \text{Ob}B \times \text{Ob}A \rightarrow \text{Ob}V \)
AND LEFT AND RIGHT ACTIONS

\[
\lambda : A (A,A') \otimes P(B,A) \rightarrow P(B,A')
\]

\[
P : P(B,A) \otimes B(B',B) \rightarrow P(B',A)
\]

SATISFYING UNIT LAWS (2) AND ASSOCIATIVITY (3).

(THINK \( V \)-FUNCTOR \( B^\circ \otimes A \rightarrow V \).

A MORPHISM OF PROFUNCTORS \( \xi : P \rightarrow Q \) IS AN EQUIVARIANT FAMILY OF MORPHISMS

\[
\xi(B,A) : P(B,A) \rightarrow Q(B,A)
\]

COMPOSITION OF PROFUNCTORS REQUIRES CERTAIN WELL-BEHAVED COLIMITS IN \( V \).
When $V = \mathbb{A}b$, a $V$-category is an additive category, and an additive category with one object is a ring.

Given $\mathbb{R}$ rings $R$, $S$, a profunctor $P : R \to S$ is an additive functor $P : S^\text{op} \times R \to \mathbb{A}b$ which is an $S$-$R$ bimodule.

Profunctor composition

$$R \xrightarrow{P} S \xrightarrow{Q} T = Q \otimes_{S} P$$

The identity is $R_{R}$.

An additive category is Cauchy complete if it has finite sums and split idempotents.
Let \((F, \varphi)_*: A \to B\) be a Mealy morphism.

Define the associated profunctor \((F, \varphi)_*\) by
\[
(F, \varphi)_*(B, A) = \varphi A \otimes B(B, FA)
\]
\[
\lambda: A(A, A') \otimes \varphi A \otimes B(B, FA) \xrightarrow{\varphi \otimes 1} \varphi (A(A, A') \otimes B(B, FA)) \xrightarrow{\lambda} \varphi A \otimes B(B, FA)
\]
\[
\beta: \varphi A \otimes B(B, FA) \otimes B(B, B') \xrightarrow{1 \otimes \beta} \varphi A \otimes B(B, FA)
\]

If \( \tau: (F, \varphi) \to (G, \varsigma) \) is a Mealy cell, define
\[
\beta_*(B, A): \varphi A \otimes B(B, FA) \xrightarrow{\lambda \otimes 1} \varphi (A(A, A') \otimes B(B, FA)) \xrightarrow{\beta} \varphi A \otimes B(B, CA).
\]

**Theorem**

1. \((F, \varphi)_*\) is a profunctor \(A \to B\).
2. \(\beta_*\) is a morphism of profunctors.
3. \((\_)_*: \text{V-Mealy}(A, B) \to \text{V-Prof}(A, B)\) is an embedding.
4. The composite \(Q \otimes (F, \varphi)_*\) exists for all profunctors \(Q: B \to C\).
5. \((G, \varsigma)_* \otimes (F, \varphi)_* = ((G, \varsigma)(F, \varphi))_*\).

"Proof" \(Q \otimes (F, \varphi)_*(C, A) = \varphi A \otimes Q(C, FA)\)
Suppose \( V \) is right closed, i.e. \((\_ \otimes V)\) has a right adjoint \([V, -]\) for every \( V \).

We will construct the **Mealy morphism classifier**:

\[
\text{Mealy morphisms } A \rightarrow B \\
\text{V-functors } A \rightarrow \hat{B}.
\]

Objects of \( \hat{B} \) are pairs \((V, B), V \in V, B \in B\)

Homs \( \hat{B}(V, B), (V', B') = [V, V' \otimes B(B, B')] \)

**Proposition** \( \hat{B} \) is a \( V \)-category

There is a Mealy morphism \((K, k) : \hat{B} \rightarrow B\)

\( K(V, B) = B \), \( k(V, B) = V \)

\( k((V, B), (V', B')) = \text{ev} : [V, V' \otimes B(B, B')] \otimes V \rightarrow V' \otimes B(B, B') \)
**Theorem** \((K, k)\) is the universal Mealy morphism, i.e., we get an equivalence of categories

\[ \mathbf{V-Cat}(A, \tilde{B}) \sim \mathbf{V-Mealy}(A, B) \]

by composing with \((K, k)\).

\(\tilde{B}\) is right adjoint to the inclusion

\(\iota_\cdot : \mathbf{V-Cat} \rightarrow \mathbf{V-Mealy}\)

\(\tilde{B}\) has another universal property:

it is the free left tensored \(\mathbf{v}\)-category generated by \(B\).

There is a fully faithful \(\mathbf{v}\)-functor

\(H : B \rightarrow \tilde{B}, \quad HB = (I, B)\).

**Theorem** \(\tilde{B}\) is left tensored and composing with \(H\) gives an equivalence of categories

\[ \mathbf{V-Ten}(\tilde{B}, C) \sim \mathbf{V-Cat}(B, C) \]