Superspans

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Octoberfest, Ottawa, 2015

Introduction

- Non-trivial intercategories?
- Intercategories $\sim$ duoidal categories with several objects
- Intercategories $\sim$ laxified double categories
- Quintessential double category is $\mathbb{S}$pan
- Taking spans in a double category produced interesting intercategories
- Identities are "too nice"
- Combine the span construction with the comma category construction
- Gives interesting and *tractable* examples of double categories

Superspans

- A *supercategory* is $\Phi : \mathbf{A} \longrightarrow \mathbf{A}^\vee$ where $\mathbf{A}^\vee$ has a choice of pullbacks
- A *superspan* (or Φ-*span*) is $\Phi A \longleftarrow B \longrightarrow \Phi \bar{A}$
- The double category $\mathbb{S}\mathrm{pan}_\Phi\, \mathbf{A}$

$$\begin{tikzcd}
A \arrow[r, "a"] & A' \\
\Phi A \arrow[r, "\Phi a"] & \Phi A' \\
B \arrow[u] \arrow[d] \arrow[r, "b"] & B' \arrow[u] \arrow[d] \\
\Phi \bar{A} \arrow[r, "\Phi \bar{a}"'] & \Phi \bar{A}' \\
\bar{A} \arrow[r, "\bar{a}"'] & \bar{A}'
\end{tikzcd}$$

- Vertical composition is pullback $\bar{B} \otimes_A B := \bar{B} \times_{\Phi A} B$
- Vertical identities are $\mathrm{id}_A = (\Phi A \longleftarrow \Phi A \longrightarrow \Phi A) = \Phi A$

Superspans (continued)

Remark

The definition only uses a choice of superpullbacks *(or* Φ-pullbacks*), i.e.*

$$\begin{tikzcd} \bar{B} \times_{\Phi A} B \arrow[r] \arrow[d] & B \arrow[d] \\ \bar{B} \arrow[r] & \Phi A \end{tikzcd}$$

but we need others later

Examples

(1) If $\Phi = 1_{\mathbf{A}}$ we get usual $\mathbb{S}\mathrm{pan}\,\mathbf{A}$

(2) If $\mathbf{B}$ has a terminal object $\top$, we can take $\mathbf{1} \xrightarrow{\top} \mathbf{B}$, $\mathbb{S}\mathrm{pan}_{\top}\,\mathbf{1}$ is the monoidal category $(\mathbf{B}, \times, \top, \ldots)$

(3) $\Phi : \mathbf{Set} \longrightarrow \mathbf{Cat}$ the "discrete functor", monads in $\mathbb{S}\mathrm{pan}_{\Phi}\,\mathbf{Set}$ are 2-categories

Colax morphisms

A *colax morphism* $(F, \phi) : \Phi \longrightarrow \Psi$ is

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "F"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\phi \Nearrow"] & \mathbf{C} \arrow[d, "\Psi"] \\
\mathbf{A}^{\vee} \arrow[r, "F^{\vee}"'] & \mathbf{C}^{\vee}
\end{tikzcd}$$

Theorem

(1) (F, ϕ) *induces a colax double functor* $\mathbb{S}\mathrm{pan}^{\phi} F : \mathbb{S}\mathrm{pan}_{\Phi} \mathbf{A} \longrightarrow \mathbb{S}\mathrm{pan}_{\Psi} \mathbf{C}$

(2) $\mathbb{S}\mathrm{pan}^{\phi} F$ *is normal if and only if* ϕ *is iso*

(3) $\mathbb{S}\mathrm{pan}^{\phi} F$ *is multiplicative if and only if*
 (a) ϕ *is monic*
 (b) $F^{\vee}$ *preserves* Φ*-pullbacks (in usual sense)*

(4) $\mathbb{S}\mathrm{pan}^{\phi} F$ *is strict if and only if*
 (a) ϕ *is an identity*
 (b) $F^{\vee}$ *preserves the choice of* Φ*-pullbacks*

"Proof"

$$\mathbb{S}\mathrm{pan}^{\phi} F : \qquad A \xrightarrow{a} A' \qquad \longmapsto \qquad FA \xrightarrow{Fa} FA'$$

$$\mathbb{S}\mathrm{pan}^{\phi} F : \qquad \begin{tikzcd} & \Phi A \\ B \arrow[ur] \arrow[dr] & \\ & \Phi \bar{A} \end{tikzcd} \qquad \longmapsto \qquad \begin{tikzcd} & & \Psi FA \\ & F^{\vee}\Phi A \arrow[ur, "\phi A"] & \\ F^{\vee}B \arrow[ur] \arrow[dr] & & \\ & F^{\vee}\Phi \bar{A} \arrow[dr, "\phi \bar{A}"'] & \\ & & \Psi F\bar{A} \end{tikzcd}$$

Conditions 3a and 3b can be reformulated in a more suggestive way as follows

Proposition

(a) ϕ monic, and
(b) $F^\vee$ preserves Φ-pullbacks
if and only if

$$\begin{tikzcd}
B \arrow[r] \arrow[d] \arrow[dr, phantom, "Pb"] & B_1 \arrow[d] \\
B_2 \arrow[r] & \Phi A
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
F^\vee B \arrow[rr] \arrow[dd] \arrow[ddrr, phantom, "Pb"] & & F^\vee B_1 \arrow[d] \\
& & F^\vee \Phi A \arrow[d, "\phi A"] \\
F^\vee B_2 \arrow[r] & F^\vee \Phi A \arrow[r, "\phi A"'] & \Psi F A
\end{tikzcd}$$

i.e. $F^\vee(B_1 \otimes_A B_2) \xrightarrow{\cong} F^\vee B_1 \otimes_{FA} F^\vee B_2$

Lax morphisms

A *lax morphism* $(F, \psi) : \Phi \longrightarrow \Psi$ is

$$\begin{tikzcd} \mathbf{A} \arrow[r, "F"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\psi \Swarrow"] & \mathbf{C} \arrow[d, "\Psi"] \\ \mathbf{A}^{\vee} \arrow[r, "F^{\vee}"'] & \mathbf{C}^{\vee} \end{tikzcd}$$

where $F^{\vee}$ preserves pullbacks

Theorem

(1) (F, ψ) induces a lax double functor $\mathbb{S}\mathrm{pan}_{\psi} F : \mathbb{S}\mathrm{pan}_{\Phi} \mathbf{A} \longrightarrow \mathbb{S}\mathrm{pan}_{\Psi} \mathbf{C}$

(2) $\mathbb{S}\mathrm{pan}_{\psi} F$ is normal if and only if ψ is monic

(3) $\mathbb{S}\mathrm{pan}_{\psi} F$ is multiplicative if and only if ψ is supercartesian

(4) If ψ is the identity, then $\mathbb{S}\mathrm{pan}_{\psi} F$ is strict if and only if $F^{\vee}$ preserves the choice of Φ-pullbacks

Construction

$$\mathbb{S}\mathrm{pan}_\psi\, F : \qquad A \xrightarrow{a} A' \qquad \longmapsto \qquad FA \xrightarrow{Fa} FA'$$

$$\mathbb{S}\mathrm{pan}_\psi\, F : \qquad \begin{tikzcd}[column sep=small] & \Phi A \\ B \arrow[ur] \arrow[dr] & \\ & \Phi\bar{A} \end{tikzcd} \qquad \longmapsto \qquad \begin{tikzcd}[column sep=small] & & \Psi FA \arrow[dr, "\psi A"] & \\ & \bullet \arrow[ur] \arrow[dr] & & F^\vee \Phi A \\ \bullet \arrow[ur] \arrow[dr] & & F^\vee B \arrow[ur] \arrow[dr] & \\ & \bullet \arrow[ur] \arrow[dr] & & F^\vee \Phi\bar{A} \\ & & \Psi F\bar{A} \arrow[ur, "\psi\bar{A}"'] & \end{tikzcd}$$

Identities

$$
\mathbb{S}\mathsf{pan}_{\psi}\, F : \quad
\begin{tikzcd}
 & \Phi A \\
\Phi A \arrow[ur, equal] \arrow[dr, equal] & \\
 & \Phi A
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
 & & \Psi FA \arrow[dl, equal] \arrow[dr, "\psi A"] & \\
 & \Psi FA \arrow[dr, "\psi A"] & & F^{\vee}\Phi A \arrow[dl, equal] \\
\ker \psi A \arrow[ur] \arrow[dr] & & F^{\vee}\Phi A \arrow[dr, equal] & \\
 & \Psi FA \arrow[ur, "\psi A"'] \arrow[dr, equal] & & F^{\vee}\Phi A \\
 & & \Psi FA \arrow[ur, "\psi A"'] &
\end{tikzcd}
$$

The unit comparison is the diagonal $\delta : \Psi FA \longrightarrow \ker \psi A$

It is an isomorphism if and only if ψA is monic

Supercartesian

Definition

ψ is *supercartesian* if (a) ψ is monic, and (b) for every $b : B \longrightarrow \Phi A$, $(F^\vee b)^{-1}(\Psi FA) \rightarrowtail F^\vee B$ is independent of A and b, i.e. for any other $\bar{b} : B \longrightarrow \Phi\bar{A}$ we have an isomorphism

$$\begin{tikzcd}
\Psi FA \arrow[d, tail, "\psi A"'] \arrow[dr, phantom, "Pb"] & \bullet \arrow[l] \arrow[d, tail] \arrow[r, phantom, "\cong"] & \bullet \arrow[r] \arrow[dl, tail] \arrow[dr, phantom, "Pb"] & \Psi F\bar{A} \arrow[d, tail, "\psi\bar{A}"] \\
F^\vee\Phi A & F^\vee B \arrow[l, "F^\vee b"] \arrow[rr, "F^\vee\bar{b}"'] & & F^\vee\Phi\bar{A}
\end{tikzcd}$$

Proposition

(1) ψ iso $\Rightarrow$ ψ supercartesian $\Rightarrow$ ψ cartesian

(2) If Φ is final then ψ supercartesian $\Leftrightarrow$ ψ cartesian

Colax choice of colimits

• A double category $\mathbb{D}$ has a *colax choice of* $\mathbf{I}$*-colimits* if the diagonal double functor $\Delta : \mathbb{D} \longrightarrow \mathbb{D}^{\mathbf{I}}$ has a left adjoint $\mathbf{I}$-colim in $\mathcal{D}bl_{\mathcal{C}lx}$

• If $\mathbb{D} = (\mathbf{D}_2 \rightrightarrows \mathbf{D}_1 \rightrightarrows \mathbf{D}_0)$ then it has a colax choice of $\mathbf{I}$-colimits if and only if $\mathbf{D}_0$ and $\mathbf{D}_1$ have a choice of $\mathbf{I}$-colimits, this choice preserved by the domain and codomain functors

Example

$$\begin{tikzcd} D_i \arrow[r, "inj_i"] \arrow[d, "x_i"'] \arrow[dr, phantom, "inj_i"] & \sum D_i \arrow[d, "\sum x_i"] \\ \bar{D}_i \arrow[r, "inj_i"'] & \sum \bar{D}_i \end{tikzcd}$$

• $\mathbb{D}$ has a *normal choice* of $\mathbf{I}$-colimits if $\mathbf{I}$-colim is normal, i.e. if $\mathbf{I}$-colimits of vertical identities are isomorphic to identities

• $\mathbb{D}$ has a *strong choice* of colimits of $\mathbf{I}$-colimits if $\mathbf{I}$-colim is strong, i.e. the vertical composite of injection cells is again an injection cell

Colimits in $\mathbb{S}\mathrm{pan}_\Phi\, \mathbf{A}$

Proposition

Let $\Phi : \mathbf{A} \longrightarrow \mathbf{A}^\vee$ be a supercategory

(1) If $\mathbf{A}$ and $\mathbf{A}^\vee$ have $\mathbf{I}$-colimits, then $\mathbb{S}\mathrm{pan}_\Phi\, \mathbf{A}$ has a colax choice of $\mathbf{I}$-colimits

(2) $\mathbf{I}$-colimits are normal if and only if Φ preserves $\mathbf{I}$-colimits

(3) $\mathbf{I}$-colimits are strong if and only if

(a) Φ preserves $\mathbf{I}$-colimits, and
(b) $\mathbf{I}$-colimits commute with Φ-pullbacks

"Proof": We have a colax morphism

$$\begin{tikzcd}
\mathbf{A}^{\mathbf{I}} \arrow[r, "colim"] \arrow[d, "\Phi^I"'] \arrow[dr, phantom, "\gamma \Nearrow"] & \mathbf{A} \arrow[d, "\Phi"] \\
\mathbf{A}^{\vee\mathbf{I}} \arrow[r, "colim"'] & \mathbf{A}^\vee
\end{tikzcd}$$

Limits in $\mathbb{S}\mathrm{pan}_\Phi\, \mathbf{A}$

Proposition

(1) If $\mathbf{A}$ *and* $\mathbf{A}^\vee$ *have* $\mathbf{I}$*-limits, then* $\mathbb{S}\mathrm{pan}_\Phi\, \mathbf{A}$ *has a lax choice of* $\mathbf{I}$*-limits*

(2) $\mathbf{I}$*-limits are normal if and only if* Φ *takes* $\mathbf{I}$*-limit cones to jointly monic families*

(3) If Φ *preserves* $\mathbf{I}$*-limits, then the* $\mathbf{I}$*-limits in* $\mathbb{S}\mathrm{pan}_\Phi\, \mathbf{A}$ *are strong*

"Proof": We have a lax morphism

$$\begin{tikzcd} \mathbf{A}^{\mathbf{I}} \arrow[r, "lim"] \arrow[d, "\Phi^I"'] \arrow[dr, phantom, "\lambda \swarrow"] & \mathbf{A} \arrow[d, "\Phi"] \\ \mathbf{A}^{\vee\mathbf{I}} \arrow[r, "lim"'] & \mathbf{A}^\vee \end{tikzcd}$$

Limits are not pointwise

E.g.

$$
\begin{tikzcd}[column sep=small]
 & \Phi A_1 \\
B_1 \arrow[ur] \arrow[dr] & \\
 & \Phi \bar{A}_1
\end{tikzcd}
\times
\begin{tikzcd}[column sep=small]
 & \Phi A_2 \\
B_2 \arrow[ur] \arrow[dr] & \\
 & \Phi \bar{A}_2
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & & \Phi(A_1 \times A_2) \arrow[dr] & \\
 & \bullet \arrow[ur] \arrow[dr] & & \Phi A_1 \times \Phi A_2 \\
\bullet \arrow[ur] \arrow[dr] & & B_1 \times B_2 \arrow[ur] \arrow[dr] & \\
 & \bullet \arrow[ur] \arrow[dr] & & \Phi \bar{A}_1 \times \Phi \bar{A}_2 \\
 & & \Phi(\bar{A}_1 \times \bar{A}_2) \arrow[ur] &
\end{tikzcd}
$$

The double category of supercategories

Objects are supercategories $\Phi : \mathbf{A} \longrightarrow \mathbf{A}^\vee$

Horizontal arrows are lax morphisms

Vertical arrows are colax morphisms

Double cells are commutative cubes

$$
\begin{tikzcd}
\mathbf{A} \arrow[rr, "F"] \arrow[dr, "\Phi"] \arrow[dd, "U"'] & & \mathbf{C} \arrow[dr, "\Psi"] \arrow[dl, phantom, "\psi\ \swarrow"] & \\
& \mathbf{A}^\vee \arrow[rr, "F^\vee"'] \arrow[dd, "U^\vee"] \arrow[dl, phantom, "\phi\ \swarrow"] & & \mathbf{C}^\vee \arrow[dd, "V^\vee"] \arrow[ddll, phantom, "\kappa^\vee\ \swarrow"] \\
\mathbf{B} \arrow[dr, "\Xi"'] & & & \\
& \mathbf{B}^\vee \arrow[rr, "G^\vee"'] & & \mathbf{D}^\vee
\end{tikzcd}
$$

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\mathbf{A} \arrow[rr, "F"] \arrow[dr, "\Phi"'] \arrow[dd, "U"'] & & \mathbf{C} \arrow[dr, "\Psi"] \arrow[dl, phantom, "\psi\,\Swarrow"] & \\
& \mathbf{A}^\vee \arrow[rr, "F^\vee"'] \arrow[dd, "U^\vee"] \arrow[ddrr, phantom, "\kappa^\vee\,\Swarrow"] & & \mathbf{C}^\vee \arrow[dd, "V^\vee"] \\
\mathbf{B} \arrow[dr, "\Xi"'] \arrow[ur, phantom, "\phi\,\Swarrow"] & & & \\
& \mathbf{B}^\vee \arrow[rr, "G^\vee"'] & & \mathbf{D}^\vee
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbf{A} \arrow[rr, "F"] \arrow[dd, "U"'] \arrow[ddrr, phantom, "\kappa\,\Swarrow"] & & \mathbf{C} \arrow[dr, "\Psi"] \arrow[dd, "V"] & \\
& & & \mathbf{C}^\vee \arrow[dd, "V^\vee"] \arrow[dl, phantom, "\Swarrow"] \\
\mathbf{B} \arrow[rr, "G"'] \arrow[dr, "\Xi"'] & & \mathbf{D} \arrow[dr, "\Sigma"'] \arrow[dl, phantom, "\Swarrow"] & \\
& \mathbf{B}^\vee \arrow[rr, "G^\vee"'] & & \mathbf{D}^\vee
\end{tikzcd}
$$

Proposition

Horizontal and vertical composition give a strict double category $\mathbb{S}$uper

The double category of double categories $\mathbb{D}\mathrm{bl}$

- Objects are weak double categories
- Horizontal arrows are lax double functors
- Vertical arrows are colax double functors
- Cells

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[d, "U"', "\bullet" marking] \arrow[dr, phantom, "\pi"] & \mathbb{C} \arrow[d, "V", "\bullet" marking] \\
\mathbb{B} \arrow[r, "G"'] & \mathbb{D}
\end{tikzcd}$$

$$\pi A : VFA \to GUA$$

$$\begin{tikzcd}
VFA \arrow[r, "\pi A"] \arrow[d, "VFv"', "\bullet" marking] \arrow[dr, phantom, "\pi v"] & GUA \arrow[d, "GUv", "\bullet" marking] \\
VF\bar{A} \arrow[r, "\pi\bar{A}"'] & GU\bar{A}
\end{tikzcd}$$

The double functor $\mathbb{S}\mathsf{pan}$

Theorem

The above constructions extend to a double functor

$$\mathbb{S}\mathsf{pan} : \mathbb{S}\mathsf{uper} \longrightarrow \mathbb{D}\mathsf{bl}$$

strict in the vertical direction and strong in the horizontal