

Superspans

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Introduction

- Non-trivial intercategories?
- Intercategories \sim duoidal categories with several objects
- Intercategories \sim laxified double categories
- Quintessential double category is $\mathbb{S}\text{pan}$
- Taking spans in a double category produced interesting intercategories
- Identities are “too nice”
- Combine the span construction with the comma category construction
- Gives interesting and *tractable* examples of double categories

Superspans

- A *supercategory* is $\Phi : \mathbf{A} \longrightarrow \mathbf{A}^\vee$ where \mathbf{A}^\vee has a choice of pullbacks
- A *superspan* (or Φ -span) is $\Phi A \longleftarrow B \longrightarrow \Phi \bar{A}$
- The double category $\text{Span}_\Phi \mathbf{A}$

$$\begin{array}{ccc} A & \xrightarrow{a} & A' \\ \Phi A & \xrightarrow{\Phi a} & \Phi A' \\ \uparrow & & \uparrow \\ B & \xrightarrow{b} & B' \\ \downarrow & & \downarrow \\ \Phi \bar{A} & \xrightarrow{\Phi \bar{a}} & \Phi \bar{A}' \\ \bar{A} & \xrightarrow{\bar{a}} & \bar{A}' \end{array}$$

- Vertical composition is pullback $\bar{B} \otimes_A B := \bar{B} \times_{\Phi A} B$
- Vertical identities are $\text{id}_A = (\Phi A \longleftarrow \Phi A \longrightarrow \Phi A) = \Phi A$

Superspans (continued)

Remark

The definition only uses a choice of superpullbacks (or Φ -pullbacks), i.e.

$$\begin{array}{ccc} \bar{B} \times_{\Phi A} B & \longrightarrow & B \\ \downarrow & & \downarrow \\ \bar{B} & \longrightarrow & \Phi A \end{array}$$

but we need others later

Examples

(1) If $\Phi = 1_{\mathbf{A}}$ we get usual $\text{Span } \mathbf{A}$

(2) If \mathbf{B} has a terminal object \top , we can take $\mathbf{1} \xrightarrow{\top} \mathbf{B}$, $\text{Span}_{\top} \mathbf{1}$ is the monoidal category $(\mathbf{B}, \times, \top, \dots)$

(3) $\Phi : \mathbf{Set} \rightarrow \mathbf{Cat}$ the “discrete functor”, monads in $\text{Span}_{\Phi} \mathbf{Set}$ are 2-categories

Colax morphisms

A *colax morphism* $(F, \phi) : \Phi \longrightarrow \Psi$ is

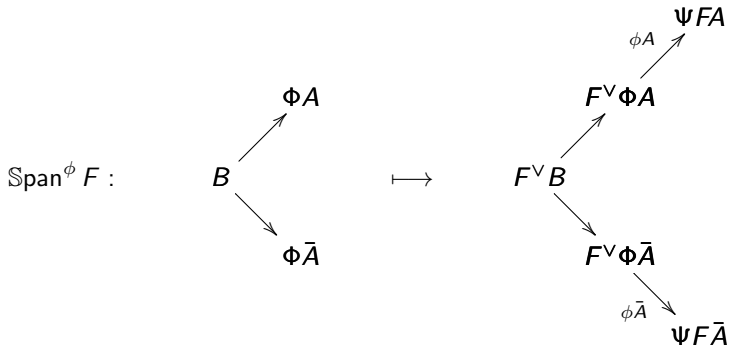
$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{F} & \mathbf{C} \\ \phi \downarrow & \nearrow \phi & \downarrow \psi \\ \mathbf{A}^\vee & \xrightarrow{F^\vee} & \mathbf{C}^\vee \end{array}$$

Theorem

- (1) (F, ϕ) induces a colax double functor $\mathbb{S}\text{pan}^\phi F : \mathbb{S}\text{pan}_\Phi \mathbf{A} \longrightarrow \mathbb{S}\text{pan}_\Psi \mathbf{C}$
- (2) $\mathbb{S}\text{pan}^\phi F$ is normal if and only if ϕ is iso
- (3) $\mathbb{S}\text{pan}^\phi F$ is multiplicative if and only if
 - (a) ϕ is monic
 - (b) F^\vee preserves Φ -pullbacks (in usual sense)
- (4) $\mathbb{S}\text{pan}^\phi F$ is strict if and only if
 - (a) ϕ is an identity
 - (b) F^\vee preserves the choice of Φ -pullbacks

“Proof”

$$\text{Span}^\phi F : \quad A \xrightarrow{a} A' \quad \longmapsto \quad FA \xrightarrow{F_a} FA'$$



Conditions 3a and 3b can be reformulated in a more suggestive way as follows

Proposition

- (a) ϕ monic, and
 (b) F^\vee preserves Φ -pullbacks
 if and only if

$$\begin{array}{ccc}
 B & \longrightarrow & B_1 \\
 \downarrow & & \downarrow \\
 B_2 & \longrightarrow & \Phi A
 \end{array}
 \xRightarrow{Pb}
 \begin{array}{ccc}
 F^\vee B & \longrightarrow & F^\vee B_1 \\
 \downarrow & & \downarrow \\
 F^\vee B_2 & \longrightarrow & F^\vee \Phi A \\
 & & \downarrow \phi_A \\
 & & \Psi F A
 \end{array}$$

i.e. $F^\vee(B_1 \otimes_A B_2) \xrightarrow{\cong} F^\vee B_1 \otimes_{F A} F^\vee B_2$

Lax morphisms

A *lax morphism* $(F, \psi) : \Phi \longrightarrow \Psi$ is

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{F} & \mathbf{C} \\ \phi \downarrow & \psi \swarrow & \downarrow \psi \\ \mathbf{A}^\vee & \xrightarrow{F^\vee} & \mathbf{C}^\vee \end{array}$$

where F^\vee preserves pullbacks

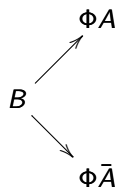
Theorem

- (1) (F, ψ) induces a lax double functor $\text{Span}_\psi F : \text{Span}_\phi \mathbf{A} \longrightarrow \text{Span}_\psi \mathbf{C}$
- (2) $\text{Span}_\psi F$ is normal if and only if ψ is monic
- (3) $\text{Span}_\psi F$ is multiplicative if and only if ψ is supercartesian
- (4) If ψ is the identity, then $\text{Span}_\psi F$ is strict if and only if F^\vee preserves the choice of Φ -pullbacks

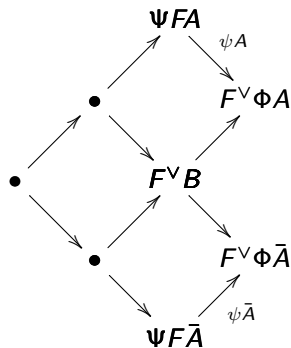
Construction

$$\text{Span}_{\psi} F : \quad A \xrightarrow{a} A' \quad \mapsto \quad FA \xrightarrow{Fa} FA'$$

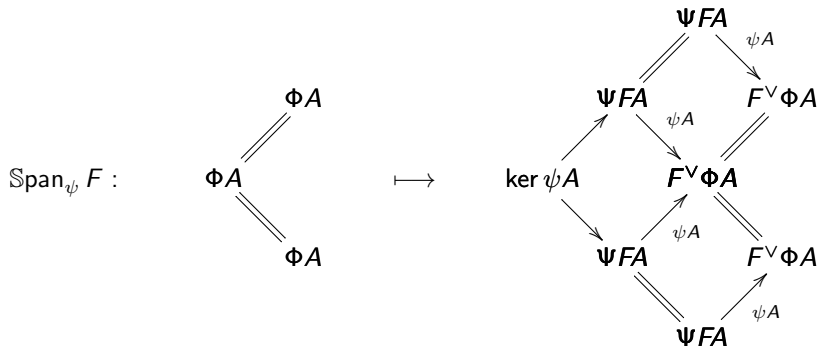
$\text{Span}_{\psi} F :$



\mapsto



Identities



The unit comparison is the diagonal $\delta : \Psi FA \longrightarrow \ker \psi A$
 It is an isomorphism if and only if ψA is monic

Supercartesian

Definition

ψ is *supercartesian* if (a) ψ is monic, and (b) for every $b : B \rightarrow \Phi A$, $(F^\vee b)^{-1}(\Psi FA) \xrightarrow{\cong} F^\vee B$ is independent of A and b , i.e. for any other $\bar{b} : B \rightarrow \Phi \bar{A}$ we have an isomorphism

$$\begin{array}{ccccc} \Psi FA & \longleftarrow & \bullet \cong \bullet & \longrightarrow & \Psi F\bar{A} \\ \downarrow \psi^A & & \downarrow & & \downarrow \psi^{\bar{A}} \\ & P_b & & P_b & \\ F^\vee \Phi A & \xleftarrow{F^\vee b} & F^\vee B & \xrightarrow{F^\vee \bar{b}} & F^\vee \Phi \bar{A} \end{array}$$

Proposition

(1) ψ iso $\Rightarrow \psi$ supercartesian $\Rightarrow \psi$ cartesian

(2) If Φ is final then ψ supercartesian $\Leftrightarrow \psi$ cartesian

Colax choice of colimits

- A double category \mathbb{D} has a *colax choice of \mathbf{I} -colimits* if the diagonal double functor $\Delta : \mathbb{D} \rightarrow \mathbb{D}^{\mathbf{I}}$ has a left adjoint $\mathbf{I}\text{-colim}$ in $\mathcal{D}bl_{\mathcal{C}lx}$
- If $\mathbb{D} = (\mathbf{D}_2 \rightrightarrows \mathbf{D}_1 \leftleftarrows \mathbf{D}_0)$ then it has a colax choice of \mathbf{I} -colimits if and only if \mathbf{D}_0 and \mathbf{D}_1 have a choice of \mathbf{I} -colimits, this choice preserved by the domain and codomain functors

Example

$$\begin{array}{ccc}
 D_i & \xrightarrow{\text{inj}_i} & \sum D_i \\
 \downarrow x_i & \text{inj}_i & \downarrow \sum x_i \\
 \bar{D}_i & \xrightarrow{\text{inj}_i} & \sum \bar{D}_i
 \end{array}$$

- \mathbb{D} has a *normal choice* of \mathbf{I} -colimits if $\mathbf{I}\text{-colim}$ is normal, i.e. if \mathbf{I} -colimits of vertical identities are isomorphic to identities
- \mathbb{D} has a *strong choice* of colimits of \mathbf{I} -colimits if $\mathbf{I}\text{-colim}$ is strong, i.e. the vertical composite of injection cells is again an injection cell

Colimits in $\text{Span}_\Phi \mathbf{A}$

Proposition

Let $\Phi : \mathbf{A} \rightarrow \mathbf{A}^\vee$ be a supercategory

(1) If \mathbf{A} and \mathbf{A}^\vee have \mathbf{I} -colimits, then $\text{Span}_\Phi \mathbf{A}$ has a colax choice of \mathbf{I} -colimits

(2) \mathbf{I} -colimits are normal if and only if Φ preserves \mathbf{I} -colimits

(3) \mathbf{I} -colimits are strong if and only if

(a) Φ preserves \mathbf{I} -colimits, and

(b) \mathbf{I} -colimits commute with Φ -pullbacks

“PROOF”: We have a colax morphism

$$\begin{array}{ccc} \mathbf{A}^{\mathbf{I}} & \xrightarrow{\text{colim}} & \mathbf{A} \\ \downarrow \Phi' & \nearrow \gamma & \downarrow \Phi \\ \mathbf{A}^{\vee \mathbf{I}} & \xrightarrow{\text{colim}} & \mathbf{A}^\vee \end{array}$$

Limits in $\text{Span}_\Phi \mathbf{A}$

Proposition

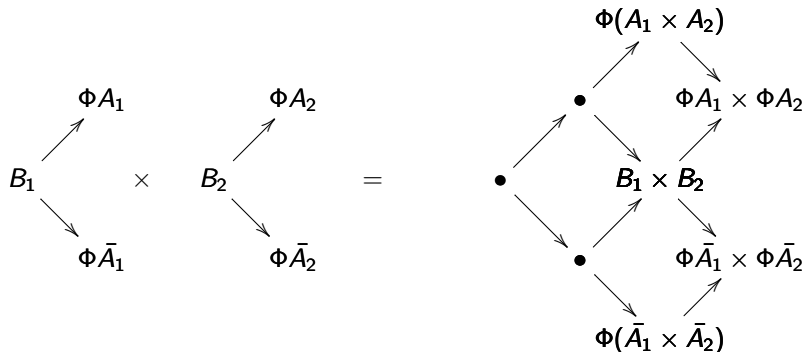
- (1) If \mathbf{A} and \mathbf{A}^\vee have \mathbf{I} -limits, then $\text{Span}_\Phi \mathbf{A}$ has a lax choice of \mathbf{I} -limits
- (2) \mathbf{I} -limits are normal if and only if Φ takes \mathbf{I} -limit cones to jointly monic families
- (3) If Φ preserves \mathbf{I} -limits, then the \mathbf{I} -limits in $\text{Span}_\Phi \mathbf{A}$ are strong

“PROOF”: We have a lax morphism

$$\begin{array}{ccc} \mathbf{A}^{\mathbf{I}} & \xrightarrow{\text{lim}} & \mathbf{A} \\ \downarrow \phi^{\mathbf{I}} & \swarrow \lambda & \downarrow \phi \\ \mathbf{A}^{\vee \mathbf{I}} & \xrightarrow{\text{lim}} & \mathbf{A}^{\vee} \end{array}$$

Limits are not pointwise

E.g.



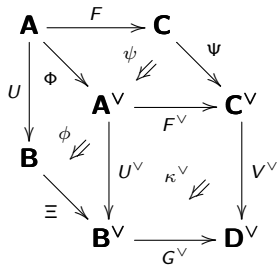
The double category of supercategories

Objects are supercategories $\Phi : \mathbf{A} \longrightarrow \mathbf{A}^\vee$

Horizontal arrows are lax morphisms

Vertical arrows are colax morphisms

Double cells are commutative cubes



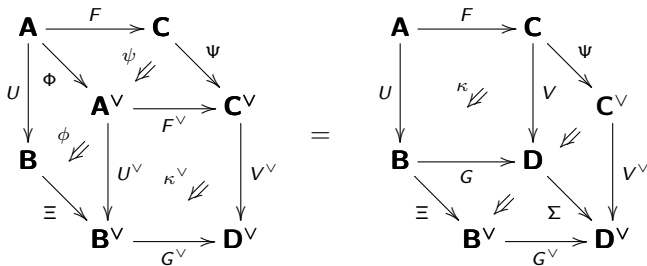
The double category of supercategories

Objects are supercategories $\Phi : \mathbf{A} \longrightarrow \mathbf{A}^\vee$

Horizontal arrows are lax morphisms

Vertical arrows are colax morphisms

Double cells are commutative cubes



Proposition

Horizontal and vertical composition give a strict double category \mathbf{Super}

The double category of double categories $\mathbb{D}bl$

- Objects are weak double categories
- Horizontal arrows are lax double functors
- Vertical arrows are colax double functors
- Cells

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{F} & \mathbb{C} \\ U \bullet \downarrow & \pi & \bullet \downarrow V \\ \mathbb{B} & \xrightarrow{G} & \mathbb{D} \end{array}$$

$$\pi A : VFA \rightarrow GUA$$

$$\begin{array}{ccc} VFA & \xrightarrow{\pi A} & GUA \\ VF_V \bullet \downarrow & \pi_V & \bullet \downarrow GU_V \\ VF\bar{A} & \xrightarrow{\pi\bar{A}} & GU\bar{A} \end{array}$$

The double functor Span

Theorem

The above constructions extend to a double functor

$$\text{Span} : \text{Super} \longrightarrow \text{DbI}$$

strict in the vertical direction and strong in the horizontal

