TILE ORDERS

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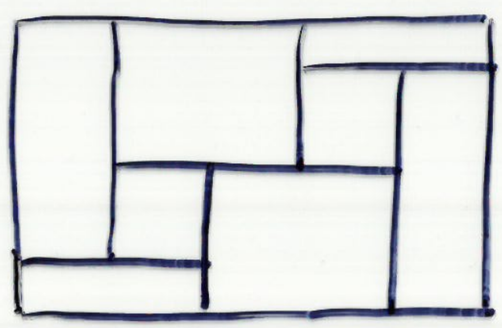
WISH TO CHARACTERIZE STRUCTURES LIKE



MOTIVATION FROM CATEGORY THY.

MORPHISMS $\bullet \longrightarrow \bullet$

COMPOSE $\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \longmapsto \bullet \xrightarrow{ab} \bullet$

ASSOCIATIVITY ALLOWS n-FOLD COMP.

$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \xrightarrow{a_3} \bullet \cdots \bullet \xrightarrow{a_n} \bullet \longmapsto \bullet \xrightarrow{a_1 a_2 a_3 \cdots a_n} \bullet$

DOUBLE CATEGORIES HAVE DOUBLE MORPHISMS

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "\Downarrow\alpha"] & \bullet \arrow[d, dotted] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

e.g. IN HOMOTOPY THEORY –

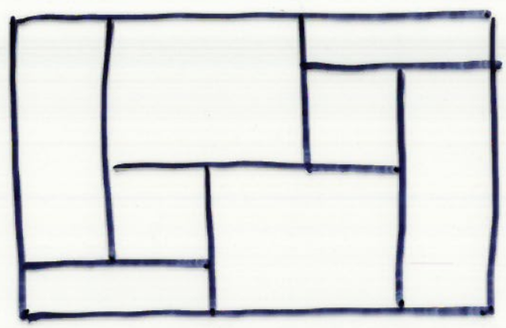
A HOMOTOPY IS $\varphi: I^2 \longrightarrow X$; $\square \mapsto$ 

2 COMPOSITIONS:

$$\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \end{array} \longmapsto \boxed{\alpha\beta}$$

$$\begin{array}{|c|}\hline \alpha \\ \hline \gamma \\ \hline \end{array} \longmapsto \boxed{\alpha\circ\gamma}$$

WHAT ABOUT



?

IN EXAMPLES IT'S THERE.

SHOULD INCORPORATE MORE GEN. COMPOSITES LIKE THIS.

GET A GENERAL ASSOCIATIVITY.

BUT WHAT ARE THE "INPUTS"?

NOT n-TUPLES ANYMORE.

WHAT IS THE ESSENTIAL STRUCTURE?

IN ONE DIMENSION :

ORDERED n-TUPLE $\longleftrightarrow$ FINITE ORDINAL

F FINITE SET ; $\leq$ BINARY REL

① <u>REFLEXIVE</u> : $\forall a \in F$, $a \leq a$

② <u>TRANSITIVE</u> : $a \leq b$ & $b \leq c \Longrightarrow a \leq c$

③ <u>ANTISYMMETRIC</u>: $a \leq b$ & $b \leq a \Longrightarrow a = b$

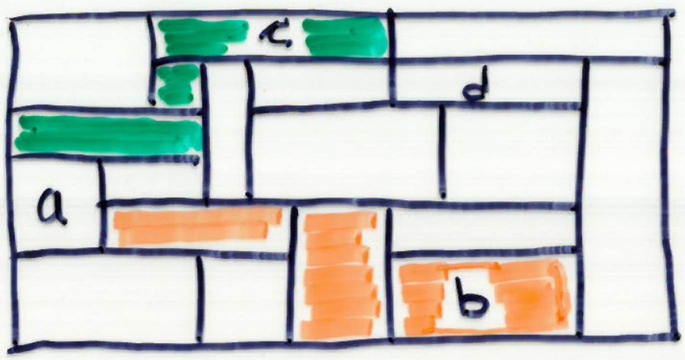
④ <u>TOTAL</u>: $\forall a, b \in F$, $a \leq b$ OR $b \leq a$

CAN REPRESENT $(F, \leq)$ AS

HIGHER MEANS BIGGER

PROOF: TAKE ANY $a \in F$. REPRESENT
$F_1 = \{x \mid x \leq a\}$ AND $F_2 = \{x \mid a < x\}$.

GIVEN A RECTANGLE DECOMPOSED INTO SMALLER RECTANGLES (TILES)



HAS 2 ORDERS

$a \leq b$ a is "to the left of" b

$\exists\ a = a_0, a_1, a_2, \cdots, a_n = b$

S.T. THE RT. BOUNDARY OF a INTERSECTS THE LEFT BOUNDARY OF b IN MORE THAN 1 PT.

$a \precsim c$ a is "below" c.

HAVE $\mathbb{T} = (T, \leq, \trianglelefteq)$ DOUBLE PARTIAL ORDER (DPO).

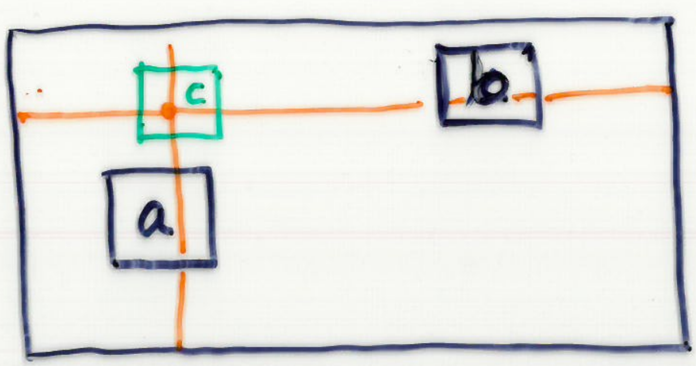
(0) TRANSITIVE, REFLEXIVE

(1) STRONG ANTISYMMETRY

ANY TWO OF $a \leq b$, $a \geq b$, $a \trianglelefteq b$, $a \trianglerighteq b$ IMPLY $a = b$.

(2) TOTALITY

$\forall a, b \ \exists c$ S.T. $a \trianglelefteq c \leq b$ OR $a \trianglerighteq c \leq b$ OR $a \trianglelefteq c \geq b$ OR $a \trianglerighteq c \geq b$.



(3) RECTANGULARITY

$\forall a \trianglelefteq c \leq b \ \exists d$ S.T. $a \leq d \trianglelefteq b$ (& DUALS)

BETTER NOTATION – REPLACE $\leq, \trianglelefteq, \ldots$

BY ARROWS $\longrightarrow, \longrightarrow\!\!\!\!\blacktriangleright, \longrightarrow\!\!\!\!\triangleright, \ldots$

RECTANGULARITY

$$\begin{tikzcd} \bullet \arrow[r, dashed] \arrow[d, dashed, -{Triangle}] & b \arrow[d, -{Triangle}] \\ a \arrow[r] & c \end{tikzcd}$$

(4) <u>BUTTERFLY PROPERTY</u>

$$\forall \begin{tikzcd} a \arrow[d] \arrow[dr, -{Triangle}] & c \arrow[d] \arrow[dl, -{Triangle[open]}] \\ b & d \end{tikzcd} \Rightarrow \exists e \begin{tikzcd} a \arrow[dd] \arrow[dr, -{Triangle}] & & c \arrow[dd] \arrow[dl, -{Triangle[open]}] \\ & e \arrow[dl, -{Triangle[open]}] \arrow[dr, -{Triangle}] & \\ b & & d \end{tikzcd}$$

THEOREM: A FINITE DPO IS A TILE ORDER IFF IT SATISFIES (0) – (4).

"PROOF" TAKE MAXIMAL $\preceq$ CHAIN

$a_1 \prec a_2 \prec a_3 \prec \cdots \prec a_n \;:\; \mathcal{C}$

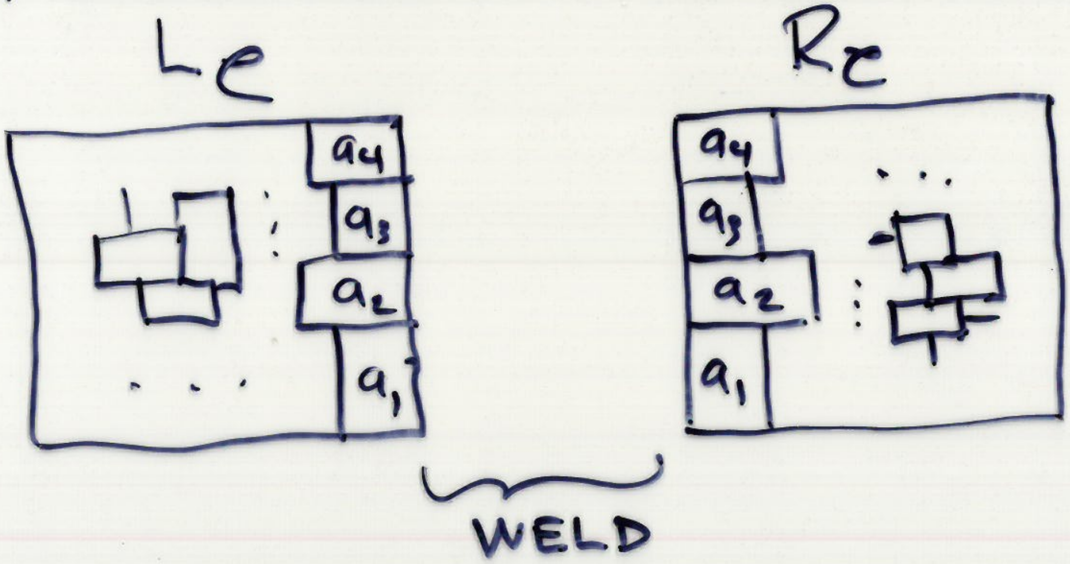
$L_{\mathcal{C}} = \{ x \mid \exists a_i \, (x \leq a_i) \}$

$R_{\mathcal{C}} = \{ x \mid \exists a_i \, (a_i \leq x) \}$

$L_{\mathcal{C}}$ & $R_{\mathcal{C}}$ IS A DPO SATISFYING (0)-(4)

USUALLY $\# L_{\mathcal{C}} < \# T$ & $\# R_{\mathcal{C}} < \# T$.

BY INDUCTION



EXCEPTIONS