Some things about double categories

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Virtual Double Category Workshop

November 28, 2022

Double categories

(Ehresmann) A *double category* is a category object in **Cat**

$$\mathbb{A}:\quad \mathbf{A}_1\times_{\mathbf{A}_0}\mathbf{A}_1 \;\begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{\bullet} \\ \xrightarrow[p_2]{} \end{array}\; \mathbf{A}_1 \;\begin{array}{c}\xrightarrow{d_0}\\ \xleftarrow{\mathrm{id}} \\ \xrightarrow[d_1]{}\end{array}\; \mathbf{A}_0$$

$$\begin{CD} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 @>{p_2}>> \mathbf{A}_1 \\ @V{p_1}VV \boxed{\text{Pb}} @VV{d_0}V \\ \mathbf{A}_1 @>>{d_1}> \mathbf{A}_0 \end{CD} \qquad\qquad \begin{CD} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 @>{\mathbf{A}_1\times_{\mathbf{A}_0}\bullet}>> \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \\ @V{\bullet\times_{\mathbf{A}_0}\mathbf{A}_1}VV \boxed{\text{Assoc}} @VV{\bullet}V \\ \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 @>>{\bullet}> \mathbf{A}_1 \end{CD}$$

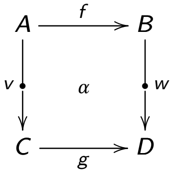
Double functor

$$\begin{array}{ccccc} \mathbf{A}_1\times_{\mathbf{A}_0}\mathbf{A}_1 & \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} & \mathbf{A}_1 & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & \mathbf{A}_0 \\ \Big\downarrow{\scriptstyle F_1\times_{F_0}F_1} & & \Big\downarrow{\scriptstyle F_1} & & \Big\downarrow{\scriptstyle F_0} \\ \mathbf{B}_1\times_{\mathbf{B}_0}\mathbf{B}_1 & \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} & \mathbf{B}_1 & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & \mathbf{B}_0 \end{array}$$

Think inside the box

$$\mathbb{A} \quad : \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\bullet}}} \mathbf{A}_1 \overset{\longrightarrow}{\underset{\longrightarrow}{\xleftarrow{\text{id}}}} \mathbf{A}_0$$

- Objects of $\mathbf{A}_0$ are *objects* of $\mathbb{A}$
- Morphisms of $\mathbf{A}_0$ are *horizontal arrows* of $\mathbb{A}$
- Objects of $\mathbf{A}_1$ are *vertical arrows* of $\mathbb{A}$
- Morphisms of $\mathbf{A}_1$ are *double cells* of $\mathbb{A}$

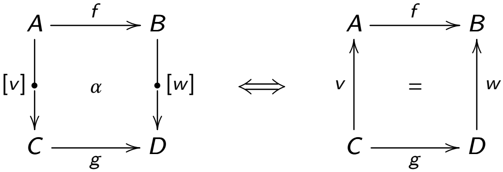


A double category is a category with two kinds of morphisms, suitably related

Opposite

A an arbitrary category

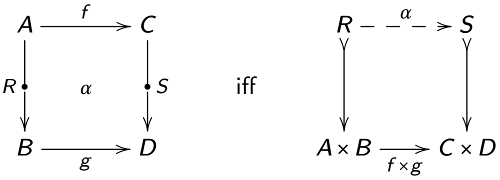
$$(\square\,\mathbf{A})^{co}$$



Student duality

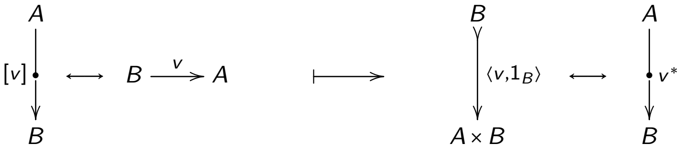
A a regular category

$\mathbb{R}\mathrm{el}(\mathbf{A})$



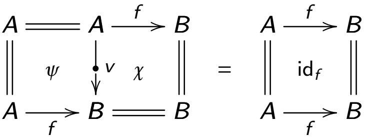
Proposition

There is a double functor $\square\mathbf{A}^{co} \longrightarrow \mathbb{R}\mathrm{el}(\mathbf{A})$ *which is the identity on objects and horizontal arrows, faithful on vertical arrows and full and faithful on cells*

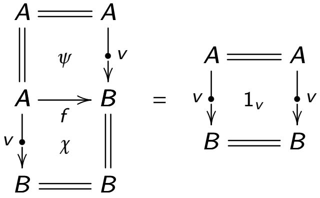


Companions

v *companion* to f



$$\chi\psi = \mathrm{id}_f$$



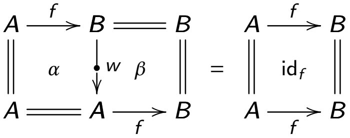
$$\chi \circ \psi = 1_v$$

Proposition

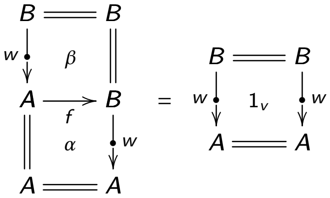
(1) If f has a companion it's unique up to isomorphism: write $v = f_*$
(2) $(1_A)_* \cong \mathrm{id}_A$
(3) $(gf)_* \cong g_* f_*$

Conjoints

w is *conjoint* to f



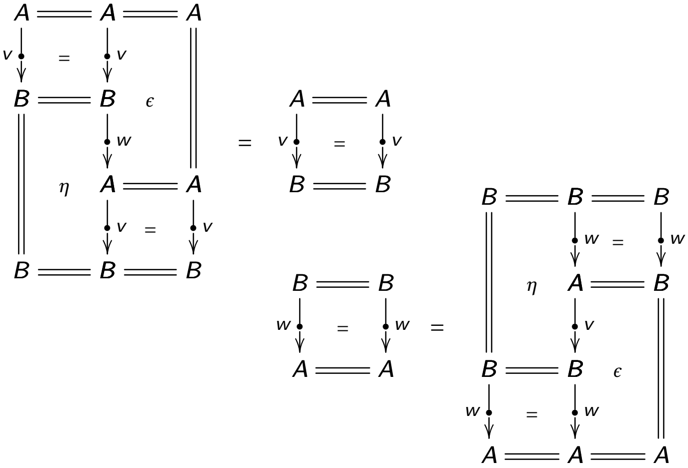
$$\beta\alpha = \mathrm{id}_f$$



$$\alpha \circ \beta = 1_w$$

- Unique up to iso: write $w = f^*$
- $1_A^* \cong \mathrm{id}_A$
- $(gf)^* \cong f^* g^*$

Adjoints



$w \dashv v$

Companions, conjoints, adjoints

Theorem

Any two of the following conditions imply the third:

(1) $v = f_*$

(2) $w = f^*$

(3) $v \dashv w$

Theorem

In $\mathbb{R}\mathrm{el}(\mathbf{A})$

(1) Every f *has a companion:* $f_* = (A \xrightarrow{\langle 1_A, f \rangle} A \times B)$

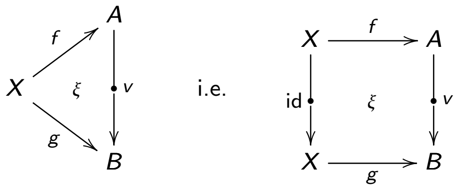
(2) Every f *has a conjoint:* $f^* = (A \xrightarrow{\langle f, 1_A \rangle} B \times A)$

(3) Every adjoint pair $R \dashv S$ *is of the form* $f_* \dashv f^*$

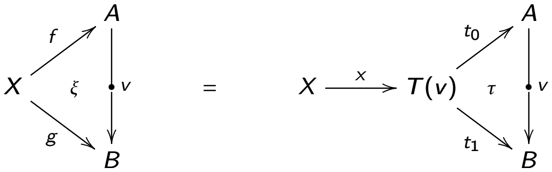
Say $\mathbb{R}\mathrm{el}(\mathbf{A})$ is *Cauchy*

Tabulators

The *tabulator* of v is a universal cell τ



$$\forall \xi \, \exists ! x (\xi = \tau x)$$

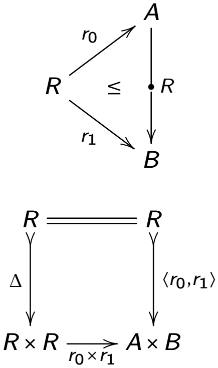


$T(v)$ is *effective* if t_1 has a companion, t_0 has a conjoint and $v \cong t_{1*} \bullet t_0^*$

Tabulating relations

Proposition

$\mathbb{R}\mathrm{el}(\mathbf{A})$ *has effective tabulators*

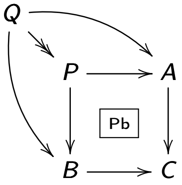


Double functors on relations

Theorem

Double functors $\mathbb{R}\mathrm{el}(\mathbf{A}) \longrightarrow \mathbb{R}\mathrm{el}(\mathbf{B})$ *"are" functors* $\mathbf{A} \longrightarrow \mathbf{B}$ *which preserve quasi-pullbacks*

Quasi-pullback Q



Transformations

$\mathbf{Doub} = \mathbf{Cat}(\mathbf{Cat})$ is cartesian closed, so $\mathbf{Doub}(\mathbb{A},\mathbb{B})$ is a double category

A *horizontal transformation* $t\colon F \longrightarrow G$

- $\forall A$ a horizontal arrow $tA\colon FA \longrightarrow GA$
- $\forall v\colon A \longrightarrow\!\!\bullet\!\!\longrightarrow A'$ a cell

$$
\begin{array}{ccc}
FA & \xrightarrow{tA} & GA \\
{\scriptstyle Fv}\big\downarrow\bullet & tv & \bullet\big\downarrow{\scriptstyle Gv} \\
FA' & \xrightarrow[tA']{} & GA'
\end{array}
$$

- Horizontally natural

$$
\begin{array}{ccccc}
FA & \xrightarrow{tA} & GA & \xrightarrow{Gf} & GC \\
{\scriptstyle Fv}\big\downarrow\bullet & tv & \bullet\big\downarrow{\scriptstyle Gv} & G\alpha & \bullet\big\downarrow{\scriptstyle Gw} \\
FA' & \xrightarrow[tA']{} & GA' & \xrightarrow[Gg]{} & GC'
\end{array}
\quad = \quad
\begin{array}{ccccc}
FA & \xrightarrow{Ff} & FC & \xrightarrow{tC} & GC \\
{\scriptstyle Fv}\big\downarrow\bullet & F\alpha & \bullet\big\downarrow{\scriptstyle Fw} & tw & \bullet\big\downarrow{\scriptstyle Gw} \\
FA' & \xrightarrow[Fg]{} & FC' & \xrightarrow[tC']{} & GC'
\end{array}
$$

Transformations (continued)

- Vertically functorial

$$
\begin{array}{ccc}
FA & \xrightarrow{tA} & GA \\
F\,\mathrm{id}_A \big\downarrow\!\bullet & t(\mathrm{id}_A) & \bullet\!\big\downarrow G\,\mathrm{id}_A \\
FA & \xrightarrow[tA]{} & GA
\end{array}
\quad = \quad
\begin{array}{ccc}
FA & \xrightarrow{tA} & GA \\
\mathrm{id}_{FA} \big\downarrow\!\bullet & \mathrm{id}_{tA} & \bullet\!\big\downarrow \mathrm{id}_{GA} \\
GA & \xrightarrow[tA]{} & GA
\end{array}
$$

$$
\begin{array}{ccc}
FA & \xrightarrow{tA} & GA \\
Fv \big\downarrow\!\bullet & tv & \bullet\!\big\downarrow Gv \\
FA' & \xrightarrow[tA']{} & GA' \\
Fv' \big\downarrow\!\bullet & tv' & \bullet\!\big\downarrow Gv' \\
FA'' & \xrightarrow[tA'']{} & GA''
\end{array}
\quad = \quad
\begin{array}{ccc}
FA & \xrightarrow{tA} & GA \\
F(v' \bullet v) \big\downarrow\!\bullet & t(v' \bullet v) & \bullet\!\big\downarrow G(v' \bullet v) \\
FA'' & \xrightarrow[tA'']{} & GA''
\end{array}
$$

Vertical transformations and cells

A *vertical transformation* $u\colon F \longrightarrow\!\!\!\!\!\!\bullet\;\; H$ is the transpose notion (switch horizontal and vertical)

- $\forall A$ a vertical arrow $uA\colon FA \longrightarrow\!\!\!\!\!\!\bullet\;\; HA$
- $\forall f\colon A \longrightarrow A'$ a cell

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ {\scriptstyle uA}\,\big\downarrow\!\bullet & uf & \bullet\!\big\downarrow\,{\scriptstyle uA'} \\ HA & \xrightarrow[Hf]{} & HA' \end{array}$$

- Vertically natural
- Horizontally functorial

A *double cell* assigns to each object A a cell νA

$$\begin{array}{ccc} F & \xrightarrow{t} & G \\ {\scriptstyle u}\,\big\downarrow\!\bullet & \nu & \bullet\!\big\downarrow\,{\scriptstyle u'} \\ H & \xrightarrow[t']{} & K \end{array} \qquad\qquad \begin{array}{ccc} FA & \xrightarrow{tA} & GA \\ {\scriptstyle uA}\,\big\downarrow\!\bullet & \nu A & \bullet\!\big\downarrow\,{\scriptstyle u'A} \\ HA & \xrightarrow[t'A]{} & KA \end{array}$$

satisfying two conditions – horizontal and vertical naturality

Transformations for $\mathbb{R}\mathrm{el}$

$F, G \colon \mathbf{A} \longrightarrow \mathbf{B}$ quasi-pullback preserving functors
$\Phi, \Psi \colon \mathbb{R}\mathrm{el}(\mathbf{A}) \longrightarrow \mathbb{R}\mathrm{el}(\mathbf{B})$ their extensions to $\mathbb{R}\mathrm{el}$

Theorem

(1) Horizontal transformations $\Phi \longrightarrow \Psi$ *are in natural bijection with natural transformations* $F \longrightarrow G$

(2) Vertical transformations $\Phi \overset{\bullet}{\longrightarrow} \Psi$ *are in natural bijection with relations*

$$V \rightarrowtail F \times G$$

in the category $QPB(\mathbf{A}, \mathbf{B})$ *of quasi-pullback preserving functors and quasi-cartesian natural transformations*

Question: Is $QPB(\mathbf{A}, \mathbf{B})$ a regular category?

Kleisli

$\mathbb{T} = (T, \eta, \mu)$ is a monad on $\mathbf{A}$

We get a double category $\mathbb{K}\mathrm{l}(\mathbb{T})$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle [v]}\,\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow\,{\scriptstyle [w]} \\ C & \xrightarrow[g]{} & D \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\,\big\downarrow & = & \big\downarrow\,{\scriptstyle w} \\ TC & \xrightarrow[Tg]{} & TD \end{array}$$

- Every horizontal arrow $f \colon A \longrightarrow B$ has a companion

$$\begin{array}{c} A \\ {\scriptstyle f_*}\,\big\downarrow\!\bullet \\ B \end{array} \quad \longleftrightarrow \quad \begin{array}{c} A \\ \big\downarrow\,{\scriptstyle f} \\ B \\ \big\downarrow\,{\scriptstyle \eta B} \\ TB \end{array} \qquad (f_* = [\eta B \cdot f])$$

- $f \colon A \longrightarrow B$ has a conjoint iff $T(f)$ iso

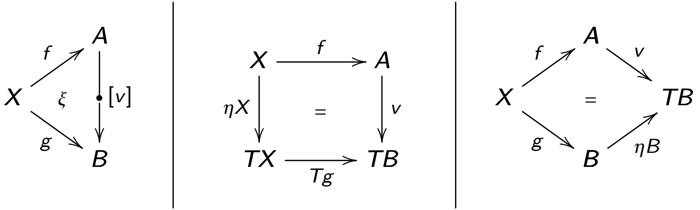
$$\begin{array}{c} B \\ {\scriptstyle f^*}\,\big\downarrow\!\bullet \\ A \end{array} \quad \longleftrightarrow \quad \begin{array}{c} B \\ \big\downarrow\,{\scriptstyle \eta B} \\ TB \\ \big\downarrow\,{\scriptstyle (Tf)^{-1}} \\ TA \end{array} \qquad (f^* = [(Tf)^{-1} \cdot \eta B])$$

Tabulating Kleisli

Proposition

$\mathbb{K}\mathrm{l}(T)$ *has tabulators iff* **A** *has pullbacks along* ηA*'s*

Proof.



□

- The tabulators are not effective

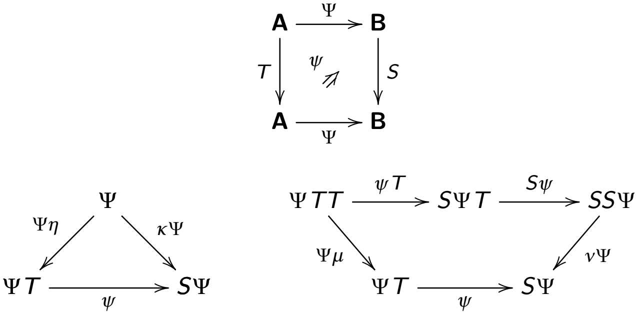
Double functors on $\mathbb{K}\mathrm{l}$

Theorem

Double functors $\mathbb{K}\mathrm{l}(\mathbb{T}) \longrightarrow \mathbb{K}\mathrm{l}(\mathbb{S})$ *correspond to monad morphisms* $\mathbb{T} \longrightarrow \mathbb{S}$

Morphism of monads:

$(\Psi, \psi) \colon \mathbb{T} \longrightarrow \mathbb{S} \qquad (\mathbb{T} = (\mathbf{A}, T, \eta, \mu),\ \mathbb{S} = (\mathbf{B}, S, \kappa, \nu))$



Transformations of monad morphisms

$(\Phi,\phi),(\Psi,\psi)\colon \mathbb{T} \longrightarrow \mathbb{S}$

A *Street 2-cell* $t\colon (\Phi,\phi) \longrightarrow (\Psi,\psi)$ is

- a natural transformation $t\colon \Phi \longrightarrow \Psi$
- satisfying

$$\begin{array}{ccc} \Phi T & \xrightarrow{tT} & \Psi T \\ {\scriptstyle\phi}\downarrow & = & \downarrow{\scriptstyle\psi} \\ S\Phi & \xrightarrow[St]{} & S\Psi \end{array}$$

Other transformations

A *Lack-Street 2-cell* $u\colon (\Phi,\phi) \longrightarrow\!\!\!\!\!\!\bullet\;\;\; (\Psi,\psi)$ is

- a natural transformation $u\colon \Phi \longrightarrow S\Psi$
- satisfying

$$
\begin{array}{ccccc}
\Phi T & & \xrightarrow{\quad\phi\quad} & & S\Phi \\
 & & & & \big\downarrow{\scriptstyle Su} \\
{\scriptstyle uT}\big\downarrow & & & & SS\Psi \\
 & & & & \big\downarrow{\scriptstyle \nu\Psi} \\
S\Psi T & \xrightarrow[S\psi]{} & SS\Psi & \xrightarrow[\nu\Psi]{} & S\Psi
\end{array}
$$

Double category version

Theorem

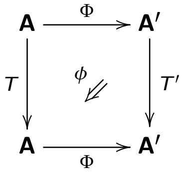
Let (Φ,ϕ) and (Ψ,ψ) be monad morphisms $\mathbb{T} \longrightarrow \mathbb{S}$ giving rise to double functors $\overline{\Phi}, \overline{\Psi} : \mathbb{K}\mathrm{l}(\mathbb{T}) \longrightarrow \mathbb{K}\mathrm{l}(\mathbb{S})$. Then

(1) horizontal transformations $\overline{\Phi} \longrightarrow \overline{\Psi}$ correspond to Street 2-cells $(\Phi,\phi) \longrightarrow (\Psi,\psi)$

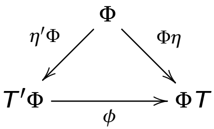
(2) vertical transformations $\overline{\Phi} \longrightarrow\!\!\!\!\!\!\bullet\;\; \overline{\Psi}$ correspond to Lack-Street 2-cells $(\Phi,\phi) \longrightarrow\!\!\!\!\!\!\bullet\;\; (\Psi,\psi)$

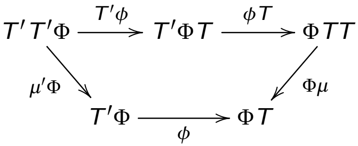
Lax morphisms of monads

- A *lax morphism of monads* (Φ, ϕ)



satisfying





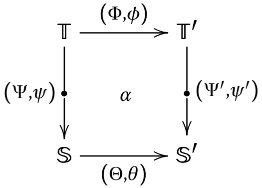
Lax vs oplax

- (Ψ, ψ) was oplax

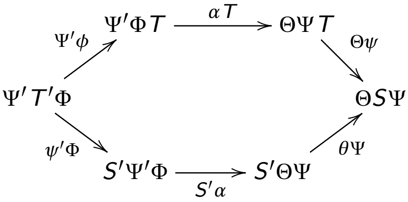
$$
\begin{array}{ccc}
\mathbf{EM}(\mathbb{T}) & \xrightarrow{\mathbf{EM}(\Phi,\phi)} & \mathbf{EM}(\mathbb{T}') \\
\downarrow & & \downarrow \\
\mathbf{A} & \xrightarrow[\Phi]{} & \mathbf{A}'
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{Kl}(\mathbb{T}) & \xrightarrow{\mathbf{Kl}(\Psi,\psi)} & \mathbf{Kl}(\mathbb{S}) \\
\uparrow & & \uparrow \\
\mathbf{A} & \xrightarrow[\Psi]{} & \mathbf{B}
\end{array}
$$

Lax and oplax together at last

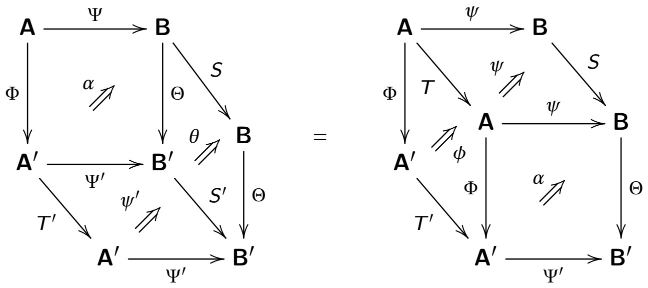
The double category $\mathbb{M}\text{onad}$



$$\Psi'\Phi \xrightarrow{\alpha} \Theta\Psi$$



Fear of hexagons



Properties of $\mathbb{M}$onad

Theorem

(1) (Φ, ϕ) *has a companion iff* ϕ *is iso*

(2) (Φ, ϕ) *has a conjoint iff* Φ *has a left adjoint*

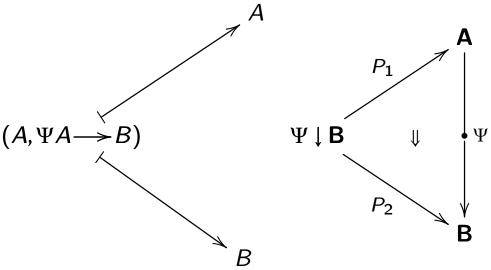
(3) $\mathbb{M}\mathrm{onad}$ *has tabulators and they are effective*

The tabulator

The tabulator of $(\Psi,\psi)\colon (\mathbf{A},T,\eta,\mu) \longrightarrow\!\!\!\!\!\!\bullet\;\; (\mathbf{B},S,\kappa,\nu)$ is given by the comma category $\Psi\downarrow\mathbf{B}$ with monad

$$\Psi\downarrow\mathbf{B} \xrightarrow{T\downarrow S} \Psi\downarrow\mathbf{B}$$

$$(A, \Psi A \xrightarrow{b} B) \quad\longmapsto\quad (TA, \Psi TA \xrightarrow{\psi A} S\Psi A \xrightarrow{Sb} SB)$$



Eilenberg-Moore for a change

A lax morphism $(\Phi, \phi)\colon \mathbb{T} \longrightarrow \mathbb{T}'$ gives an algebraic functor over Φ

$$(TA \xrightarrow{a} A) \longmapsto (T'\Phi A \xrightarrow{\phi A} \Phi TA \xrightarrow{\Phi a} \Phi A)$$

$$\begin{array}{ccc}
\mathbf{EM}(\mathbb{T}) & \xrightarrow{\mathbf{EM}(\Phi,\phi)} & \mathbf{EM}(\mathbb{T}') \\
\big\downarrow & & \big\downarrow \\
\mathbf{A} & \xrightarrow[\Phi]{} & \mathbf{A}'
\end{array}$$

But what about oplax morphisms $(\Psi, \psi)\colon \mathbb{T} \mathrel{-\!\!\bullet\!\!\to} \mathbb{S}$?

Profunctors make a cameo appearance

$$\mathbf{EM}(\Psi,\psi)\colon \mathbf{EM}(\mathbb{T}) \xrightarrow{\bullet} \mathbf{EM}(\mathbb{S})$$

$$\mathbf{EM}(\Psi,\psi)\colon \mathbf{EM}(\mathbb{T})^{op} \times \mathbf{EM}(\mathbb{S}) \longrightarrow \mathbf{Set}$$

An element of $\mathbf{EM}(\Psi,\psi)((A,a),(B,b))$ is $x\colon \Psi A \longrightarrow B$

$$\begin{array}{ccccc}
\Psi TA & \xrightarrow{\psi A} & S\Psi A & \xrightarrow{Sx} & SB \\
{\scriptstyle \Psi a}\downarrow & & & & \downarrow{\scriptstyle b} \\
\Psi A & & \xrightarrow[x]{\qquad\qquad} & & B
\end{array}$$

EM extends to cells in $\mathbb{M}\mathrm{onad}$

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{(\Phi,\phi)} & \mathbb{T}' \\ {\scriptstyle(\Psi,\psi)}\big\downarrow & \alpha & \big\downarrow{\scriptstyle(\Psi',\psi')} \\ \mathbb{S} & \xrightarrow[(\Theta,\theta)]{} & \mathbb{S}' \end{array}$$

$$\begin{array}{ccc} \mathbf{EM}(\mathbb{T}) & \xrightarrow{\mathbf{EM}(\Phi,\phi)} & \mathbf{EM}(\mathbb{T}') \\ {\scriptstyle(\Psi,\psi)}\big\downarrow & \overset{\mathbf{EM}(\alpha)}{\Longrightarrow} & \big\downarrow{\scriptstyle(\Psi',\psi')} \\ \mathbf{EM}(\mathbb{S}) & \xrightarrow[\mathbf{EM}(\Theta,\theta)]{} & \mathbf{EM}(\mathbb{S}') \end{array}$$

$$\mathbf{EM}(\alpha)\colon (\Psi A \xrightarrow{x} B) \quad \longmapsto \quad (\Psi'\Phi A \xrightarrow{\alpha A} \Theta\Psi A \xrightarrow{\Theta x} \Theta B)$$

$$\mathbf{EM}\colon \mathbb{M}\mathrm{onad} \longrightarrow \mathbb{C}\mathrm{at}$$

$\mathbb{C}\mathrm{at}$

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{F} & \mathbf{B} \\ {\scriptstyle P}\big\downarrow\!\bullet & \overset{t}{\Rightarrow} & \bullet\!\big\downarrow{\scriptstyle Q} \\ \mathbf{C} & \xrightarrow[G]{} & \mathbf{D} \end{array}$$

- $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ categories
- F, G functors
- P, Q profunctors

$$P\colon \mathbf{A}^{op} \times \mathbf{C} \longrightarrow \mathbf{Set}\ , Q\colon \mathbf{B}^{op} \times \mathbf{D} \longrightarrow \mathbf{Set}$$

- t natural transformation

$$t\colon P(-,=) \longrightarrow Q(F-, G=)$$

- Composition of profunctors uses coends, and is not associative on the nose

$\mathbb{C}\mathrm{at}$ is a *weak double category*
$\mathbf{EM}\colon \mathbb{M}\mathrm{onad} \longrightarrow \mathbb{C}\mathrm{at}$ is a *lax double functor*

Double categories

(Ehresmann) A *double category* is a category object in **Cat**

$$\mathbb{A}\colon \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \overset{p_1}{\underset{p_2}{\overset{\bullet}{\rightrightarrows}}} \mathbf{A}_1 \overset{d_0}{\underset{d_1}{\overset{\text{id}}{\leftrightarrows}}} \mathbf{A}_0$$

$$\begin{array}{ccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{p_2} & \mathbf{A}_1 \\ {\scriptstyle p_1}\downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle d_0} \\ \mathbf{A}_1 & \xrightarrow[d_1]{} & \mathbf{A}_0 \end{array} \qquad \begin{array}{ccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{\mathbf{A}_1 \times_{\mathbf{A}_0} \bullet} & \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \\ {\scriptstyle \bullet \times_{\mathbf{A}_0} \mathbf{A}_1}\downarrow & \boxed{\text{Assoc}} & \downarrow{\scriptstyle \bullet} \\ \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow[\bullet]{} & \mathbf{A}_1 \end{array}$$

Double functor

$$\begin{array}{ccccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \rightrightarrows & \mathbf{A}_1 & \leftrightarrows & \mathbf{A}_0 \\ {\scriptstyle F_1 \times_{F_0} F_1}\downarrow & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\ \mathbf{B}_1 \times_{\mathbf{B}_0} \mathbf{B}_1 & \rightrightarrows & \mathbf{B}_1 & \leftrightarrows & \mathbf{B}_0 \end{array}$$

Weak double categories

A *(weak) double category* is a weak category object in $\mathscr{C}at$

$$\mathbb{A}\colon \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \;\overset{p_1}{\underset{p_2}{\overset{\bullet}{\rightrightarrows}}}\; \mathbf{A}_1 \;\overset{d_0}{\underset{d_1}{\overset{\xleftarrow{\ \mathrm{id}\ }}{\rightrightarrows}}}\; \mathbf{A}_0$$

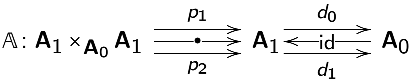
$$\begin{array}{ccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{p_2} & \mathbf{A}_1 \\ {\scriptstyle p_1}\downarrow & \boxed{\mathrm{Pb}} & \downarrow{\scriptstyle d_0} \\ \mathbf{A}_1 & \xrightarrow[d_1]{} & \mathbf{A}_0 \end{array} \qquad \begin{array}{ccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{\mathbf{A}_1 \times_{\mathbf{A}_0} \bullet} & \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \\ {\scriptstyle \bullet \times_{\mathbf{A}_0} \mathbf{A}_1}\downarrow & \overset{\alpha}{\swarrow}{\scriptstyle \cong} & \downarrow{\scriptstyle \bullet} \\ \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow[\bullet]{} & \mathbf{A}_1 \end{array}$$

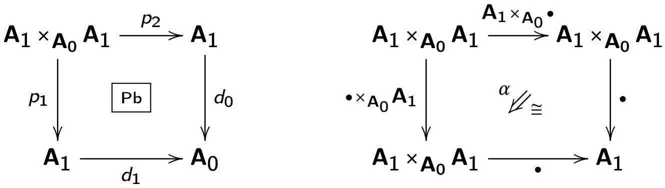
Double functor

$$\begin{array}{ccccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \rightrightarrows & \mathbf{A}_1 & \rightleftarrows & \mathbf{A}_0 \\ {\scriptstyle F_1 \times_{F_0} F_1}\downarrow & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\ \mathbf{B}_1 \times_{\mathbf{B}_0} \mathbf{B}_1 & \rightrightarrows & \mathbf{B}_1 & \rightleftarrows & \mathbf{B}_0 \end{array}$$

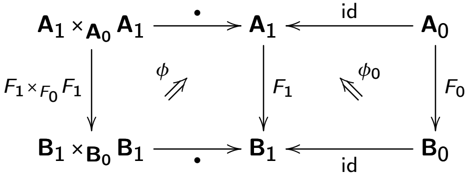
Lax double functors of weak double categories

A *(weak) double category* is a weak category object in $\mathscr{C}at$





Lax double functor



Full circle

And, this is where the story begins...

Thank you!