MAT 1341B, INTRODUCTION TO LINEAR ALGEBRA, FALL 2003

Answers to First Midterm, October 3, 2003

Prof. P. Selinger

Question 1. Which of the following statements are true?

I. The span of two non-zero vectors u and v in $\mathbb{R}^3$ is always a plane through the origin.

Answer: *No, it could be a line, if the two vectors are collinear.*

II. The span of a single non-zero vector in $\mathbb{R}^2$ is always a line.

Answer: *Yes.*

III. A set of vectors $\{u, v, w\} \subseteq \mathbb{R}^3$ is a spanning set of $\mathbb{R}^3$ if every $x \in \mathbb{R}^3$ is a linear combination of u, v, and w.

Answer: *Yes, indeed, this is the definition of a spanning set.*

IV. $\{(1,0,1),(1,2,0),(0,2,-1)\}$ spans $\mathbb{R}^3$.

Answer: *No, they don't. Note that* $(0,2,-1) = (1,2,0) - (1,0,1)$, *so the span of these three vectors is a plane.*

V. $\{t^2+1, 2t+1, t^2+2t\}$ spans $P_2(t)$.

Answer: *Yes. Let* at^2+bt+c *be an arbitrary element in* $P_2(t)$. *We are looking for* k, l, m *such that* $at^2+bt+c = k(t^2+1)+l(2t+1)+m(t^2+2t)$. *This is equivalent to*

$$\begin{array}{rcl} a & = & k+m \\ b & = & 2l+2m \\ c & = & k+l \end{array}$$

We solve and find $k = \frac{1}{2}a - \frac{1}{4}b + \frac{1}{2}c$, $-\frac{1}{2}a + \frac{1}{4}b + \frac{1}{2}c$, *and* $m = \frac{1}{2}a + \frac{1}{4}b - \frac{1}{2}c$.

Question 2. Which of the following are vector spaces?

(1) $\{(x,y,z) \in \mathbb{R}^3 \mid x+2y-z=0\}$, with the usual vector operations of $\mathbb{R}^3$.

Answer: *Yes.*

(2) $\{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=1\}$, with the usual vector operations of $\mathbb{R}^3$.

Answer: *No, because* $(0,0,0) \notin V$.

(3) $\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid b = c$ and $d = 0\}$, with the usual vector operations of $M_{2,2}$.

Answer: *Yes.*

Question 3. Let $V = P_2(t)$, and let $u = 3t^2+2t+1$ and $v = 2t^2-t+1$. Which of the following vectors are linear combinations of u and v?

$w_1 = 7t^2+3$.

Answer: *Yes:* $w_1 = u+2v$.

$w_2 = t^2+3t$.

Answer: *Yes:* $w_2 = u - v$.

$w_3 = 4t^2 + 3t + 2$.

Answer: *No: if we set* $w_3 = au + bv$, *we get* $4t^2 + 3t + 2 = a(3t^2 + 2t + 1) + b(2t^2 - t + 1)$, *therefore*

$$\begin{bmatrix} 3a+2b & = & 4 \\ 2a-b & = & 3 \\ a+b & = & 2 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} a+b & = & 2 \\ 0a-b & = & -2 \\ 0a-3b & = & -1 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} a+b & = & 2 \\ 0a-b & = & -2 \\ 0a-0b & = & 5 \end{bmatrix},$$

a contradiction.

Question 4. Consider the following linear system of equations:

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 4 \\ 2x & + & y & + & 5z & = & -1 \\ x & + & y & + & 2z & = & 1 \end{array}$$

After reducing this system to Echelon form, there are:

A. 2 pivot variables and 1 free variable.

B. 3 free variables.

C. 1 pivot variable and 2 free variables.

D. 3 pivot variables.

E. The system is inconsistent.

F. The system cannot be reduced to Echelon form.

Answer: *We reduce the system to Echelon form:*

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 1 & 5 & -1 \\ 1 & 1 & 2 & 1 \end{array}\right] \Longleftrightarrow_{\substack{L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 2L_1}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 3 & -9 \\ 0 & -1 & 1 & -3 \end{array}\right] \Longleftrightarrow_{L_3 \leftarrow L_3 - \frac{1}{3}L_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 3 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the correct answer is: there are 2 pivot variables and 1 free variable.

Question 5. Consider the vector space $\mathbf{F}(\mathbb{R}) = \{\mathbf{f} \mid \mathbf{f} : \mathbb{R} \to \mathbb{R}\}$, with the standard operations. Let $W = \{f \in \mathbf{F}(\mathbb{R}) \mid f(-x) = -f(x) \text{ for all } x \in \mathbb{R}\}$.

(a) Which of the following functions are in W? Justify your answer.

$$f_1(x) = x^2,$$
$$f_2(x) = x^3,$$
$$f_3(x) = \sin(x),$$
$$f_4(x) = 0.$$

Answer: *$f_1 \notin W$ because $(-x)^2 \neq -(x^2)$ in general, for instance, let $x = -1$. $f_2 \in W$ because $-(x^3) = (-x)^3$ for all $x \in \mathbb{R}$. $f_3 \in W$ because $sin(-x) = -sin(x)$ for all $x \in \mathbb{R}$. $f_4 \in W$ because $f_4(-x) = 0 = -0 = -f_4(x)$ for all $x \in \mathbb{R}$.*

(b) Prove that W is a subspace of $\mathbf{F}(\mathbb{R})$.

Answer: *(1) we know $0 \in W$ because it was shown in part (a).*

(2) Assume $f, g \in W$. We have to show that $f + g \in W$. By assumption, we have $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all x. Therefore, we have $(f + g)(-x) = f(-x) + g(-x) = -f(x) + (-g(x)) = -(f(x) + g(x)) = -(f + g)(x)$ for all x (using the definition of $f + g$, the hypothesis, and properties of addition of real numbers). It follows that $f + g \in W$.

(3) Assume $f \in W$ and $k \in \mathbb{R}$. We have to show that $kf \in W$. By assumption, we have $f(-x) = -f(x)$ for all x. Therefore, we have $(kf)(-x) = k(f(-x)) = k(-f(x)) = -k(f(x)) = -(kf)(x)$, using the definition of kf, the hypothesis, and properties of multiplication of real numbers. It follows that $kf \in W$.

Question 6. Let $V = \mathbb{R}^2$, and consider the following "addition" and "scalar multiplication" operation on V, where $(x, y), (z, w) \in \mathbb{R}^2$ and $k \in \mathbb{R}$:

$$\begin{aligned} (x, y) \oplus (z, w) &= (x + z + 1, y + w + 1) \\ k \odot (x, y) &= (kx, ky) \end{aligned}$$

Recall the vector space axioms:

[A$_1$] $(u + v) + w = u + (v + w)$.

[A$_2$] There is a zero vector $0 \in V$ such that $u + 0 = u = 0 + u$.

[A$_3$] For each $u \in V$ there is $-u \in V$ such that $u + (-u) = 0 = (-u) + u$.

[A$_4$] $u + v = v + u$.

[M$_1$] $k(u + v) = ku + kv$.

[M$_2$] $(a + b)u = au + bu$.

[M$_3$] $(ab)u = a(bu)$.

[M$_4$] $1u = u$.

(a) Prove that [A$_1$] holds with respect to the operation $\oplus$.

Answer: *Let $u = (x, y)$, $v = (x', y')$, and $w = (x'', y'')$. Then*

$$\begin{aligned} (u \oplus v) \oplus w &= ((x, y) \oplus (x', y')) \oplus (x'', y'') \\ &= (x + x' + 1, y + y' + 1) \oplus (x'', y'') \\ &= (x + x' + 1 + x'' + 1, y + y' + 1 + y'' + 1) \end{aligned}$$

and

$$\begin{aligned} u \oplus (v \oplus w) &= (x, y) \oplus ((x', y') \oplus (x'', y'')) \\ &= (x, y) \oplus (x' + x'' + 1, y' + y'' + 1) \\ &= (x + x' + x'' + 1 + 1, y + y' + y'' + 1 + 1) \end{aligned}$$

Since the left- and right-hand sides are equal, the property holds.

(b) Prove that [A$_2$] holds. Which vector will be the "zero" vector for the operation $\oplus$?

Answer: *We let* $\mathbf{0} = (-1,-1)$. *Then we have:*

$$\begin{aligned} u \oplus \mathbf{0} &= (x,y) \oplus (-1,-1) \\ &= (x-1+1, y-1'+1) \\ &= (x,y) \\ &= u \end{aligned}$$

and similarly for $\mathbf{0} \oplus u = u$.

(c) Prove that [M$_2$] does not hold with respect to the operations $\oplus$ and $\odot$, by giving a *concrete counterexample*.

Answer: *Let* $a = 0$, $b = 0$, *and* $u = (0,0)$. *Then* $(a+b)u = 0u = (0,0)$, *whereas* $au \oplus bu = 0u \oplus 0u = (0,0) \oplus (0,0) = (1,1)$. *So* $(a+b)u \neq au \oplus bu$ *and* M_2 *fails.*

Question 7. Consider the following system of linear equations.

$$\begin{array}{rcrcrcrcr} 1x & + & 1y & + & 3z & + & 2w & = & -3 \\ 2x & + & 0y & + & 2z & + & 1w & = & -1 \\ 3x & + & -1y & + & 1z & + & 2w & = & 3 \end{array}$$

Show all your work and justify your answers. Remember to check your answers for correctness!

(a) Find the general solution of this system.

Answer:

$$\left[\begin{array}{rrrr|r} 1 & 1 & 3 & 2 & -3 \\ 2 & 0 & 2 & 1 & -1 \\ 3 & -1 & 1 & 2 & 3 \end{array}\right] \Longleftrightarrow_{\substack{L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1}} \left[\begin{array}{rrrr|r} 1 & 1 & 3 & 2 & -3 \\ 0 & -2 & -4 & -3 & 5 \\ 0 & -4 & -8 & -4 & 12 \end{array}\right]$$

$$\Longleftrightarrow_{\substack{L_3 \leftarrow -\frac{1}{4}L_3 \\ L_2 \leftrightarrow L_3}} \left[\begin{array}{rrrr|r} 1 & 1 & 3 & 2 & -3 \\ 0 & 1 & 2 & 1 & -3 \\ 0 & -2 & -4 & -3 & 5 \end{array}\right]$$

$$\Longleftrightarrow_{L_3 \leftarrow L_3 + 2L_2} \left[\begin{array}{rrrr|r} 1 & 1 & 3 & 2 & -3 \\ 0 & 1 & 2 & 1 & -3 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right]$$

Using back substitution, we find

$$\begin{aligned} &w = 1 \\ &z = a \\ &y = -3 - 2z - w = -3 - 2a - 1 = -4 - 2a \\ &x = -3 - y - 3z - 2w = -3 + 4 + 2a - 3a - 2 = -1 - a \end{aligned}$$

or equivalently, $(x,y,z,w) = (-1,-4,0,1) + a(-1,-2,1,0)$.

(b) Find a solution (x, y, z, w) such that $z = 4$.

Answer: *Using the general solution from (a), we find that* $z = 4$ *if* $a = 4$*, thus* $(x, y, z, w) = (-1, -4, 0, 1) + 4(-1, -2, 1, 0) = (-5, -12, 4, 1)$.

(c) Does there exist a solution with $w = 2$?

Answer: *No, because* $w = 1$ *in the general solution.*

Question 8. (a) Write the polynomial $f(t) = t^2 + 2t + 3$ as a linear combination of $p_1(t) = t^2$, $p_2(t) = (t+1)^2$, and $p_3 = (t+2)^2$.

Answer: *We want* $f(t) = ap_1(t) + bp_2(t) + cp_3(t)$*, or equivalently,* $t^2 + 2t + 3 = at^2 + b(t^2 + 2t + 1) + c(t^2 + 4t + 4)$*. This translates into* $a + b + c = 1$*,* $2b + 4c = 2$*,* $b + 4c = 3$*. Subtracting the last two equations from each other, we find* $b = -1$*, therefore* $c = 1$*, and finally* $a = 1$*. Thus,*

$$f(t) = p_1(t) - p_2(t) + p_3(t).$$

(b) Find the value of a such that $v = (1, 3, a)$ can be written as a linear combination of $w = (1, 1, 1)$ and $u = (1, -1, -2)$.

Answer: *We want* k, l *such that* $kw + lu = v$*. This means,* $k(1, 1, 1) + l(1, -1, -2) = (1, 3, a)$*. By looking at each coordinate, we obtain three equations:* $k + l = 1$*,* $k - l = 3$*, and* $k - 2l = a$*. Solving the first two equations for* k *and* l*, we find* $k = 2$*,* $l = -1$*. Therefore* $a = k - 2l = 4$*.*