7. Consider the linear system

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 0 \\ x & + & 3y & + & 2kz & = & 0 \\ 2x & + & 4y & + & kz & = & 0 \end{array}$$

a) If A is the coefficient matrix of the system above, find $\operatorname{rank} A$ for all values of k.

b) Find all k so that this system has

  i) a unique solution,

  ii) infinitely many solutions, and

  iii) no solutions.

c) In case (ii) above, give a geometric description of the set of solutions in each instance.

8. a) A pet fanatic, Pat, wishes to keep combination of dogs, cats and weasels. They are to be fed 3 food types, A, B and C, and their weekly requirements (in kgs per animal per week) are given in the table below:

	cats	dogs	weasels
A	2	0	2
B	2	1	1
C	3	1	2

Pat can only afford 12 kgs of A, 10 kgs of B and 16 kgs of C each week. Assuming that all the available food is eaten by Pat's furry friends, write down a linear system, <u>together with all constraints</u>, describing the possible combinations of cats, dogs and weasels that this lunatic can keep. Don't forget to define your variables and **DO NOT SOLVE THIS SYSTEM.**

b) Find all solutions of the following constrained linear system:

$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 14 \\ 3x & + & 3y & & & = & 15 \\ 5x & + & 4y & + & z & = & 29 \end{array}$$

where x, y and z are integers, and $x \geq 3$, $y \geq 0$ and $x \geq 2$.

9. Let $W = \{(a, a-b, a+b) \mid a, b \in \mathbf{R}\}$.

a) By any method, show that W is a subspace of $\mathbb{R}^3$.
b) Find a basis of W and give the dimension of W. (You must explain why your choice is a basis.)
c) Give a geometric description of W.
d) Give an equation for W.

10. Consider the vector space

$$\mathbf{F}[-1,1] = \{f \mid f \text{ is a real-valued function with domain } [-1,1]\},$$

together with the usual operations on functions. Recall that the zero of $\mathbf{F}[-1,1]$ is the function that has the value 0 for every $x \in [-1,1]$.

a) Prove that $\{1, x, x^2\}$ is linearly independent in $\mathbf{F}[-1,1]$.
(Hint: Consider $x = -1, 0$, and 1.)

b) If $f(x) = (2x+1)^2$, is $f \in \operatorname{span}\{1, x, x^2\}$?

c) Show that if $g(x) = \frac{1}{x+2}$, then $g \notin \operatorname{span}\{1, x\}$

d) What is the dimension of the subspace $U = \operatorname{span}\{1, x, \frac{1}{x+2}\}$?

11. a) Let A be a real $n \times n$ matrix. Give 3 different statements which are equivalent to:

$$\det A \neq 0.$$

b) Say whether the following statements are TRUE or FALSE. No justification is necessary!

i) If is A a real $n \times n$ matrix, then A is invertible iff 0 is not an eigenvalue of A.
ii) A 3×4 matrix can have linearly independent columns.
iii) The solution space of a homogeneous system of 2 equations in 6 unknowns can have dimension exactly 3.

12. Let $A = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. The eigenvalues of A are 0 and 6.

a) Find a basis of $E_0 = \{x \in \mathbf{R}^3 \mid Ax = 0\}$.

b) Find a basis of $E_6 = \{x \in \mathbf{R}^3 \mid (A - 6I)x = 0\}$.

c) Show that the set consisting of all vectors in the bases for E_0 and E_6 is a basis for $\mathbf{R}^3$.

d) If possible, find an invertible matrix P such that $P^{-1}AP = D$ is diagonal and give this diagonal matrix D.