7. A Norweigan Blue parrot is advised by a nutritionist to take 14 units of vitamin A, 15 units of vitamin D and 29 units of vitamin E each day. The parrot can choose from three brands I, II and III, and the amount of each vitamin in every capsule of the various brands is given below:

	I	II	III
vitamin A	2	1	1
vitamin D	3	3	0
vitamin E	5	4	1

a) Write down a system of equations and constraints that determine the possible combinations of numbers of capsules of each brand that will provide exactly the required amounts of vitamins for the parrot. Don't forget to define your variables.

b) Solve the constrained linear system of (a).

c) If the respective costs per capsule of brands I, II and II are 30, 15 and 12, determine the choice which will minimize the total cost.

8. Let $S = \{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\}$.

a) Show that S is linearly dependent.

b) Explain why $\operatorname{span} S = \operatorname{span}\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\}$

c) Find the dimension of $\operatorname{span} S$.

d) Is $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \operatorname{span} S$?

9. Consider the 2 planes $\{(x, y, z) \mid x - 3y + 4z = 0\}$ and $\{(x, y, z) \mid 2x - 7y + 9z = 0\}$.

a) If V is the intersection of these planes, give a geometric despcription of V, and explain why V is a subspace of $\mathbf{R}^3$. (You do not need to use the subspace test here.)

b) Find a basis of V.

c) If $v = n_1 \times n_2$ is the cross product of the normals n_1 and n_2 of the planes above, show that $V = \text{span}\{v\}$.

d) Show that $\{n_1, n_2, v\}$ is a basis of $\mathbf{R}^3$. Is this an orthogonal basis ?

10. Let $P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

a) Show that $P^2 = P$.

b) Find a basis for the column space $\mathrm{col}(P)$ of P, i.e. the subspace of $\mathbf{R}^3$ spanned by the columns of P. Hence, determine $\dim \mathrm{col}(P)$ and compare it with $\mathrm{rank}\, P$.

c) Explain why $\det P = 0$ <u>without computing</u> $\det P$. (Use a theorem!)

d) If Q is *any* 3 by 3 matrix satisfying $Q^2 = Q$, show that 0 or 1 are the only possible eigenvalues of Q.

11. Let $A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$. The eigenvalues of A are -2 and 4.

a) Find a basis of $E_{-2} = \{x \in \mathbf{R}^3 \mid (A + 2I)x = 0\}$.

b) Find a basis of $E_4 = \{x \in \mathbf{R}^3 \mid (A - 4I)x = 0\}$.

c) Show that the set consisting of all vectors in the bases for E_{-2} and E_4 is a basis for $\mathbf{R}^3$.

d) If possible, find an invertible matrix P such that $P^{-1}AP = D$ is diagonal and give this diagonal matrix D.

12. State whether the following are always true or could be false. If true, explain why, if false, give a numerical example to illustrate.

a) If a 3 by 5 matrix A has rank 3, the system $Ax = 0$ has a unique solution.

b) If $\{u, v, w\}$ are 3 vectors in a vector space V, then $\{u, v, w\}$ is linearly independent if and only if $au + bv + cw = 0$ where the scalars a, b and c satisfy $a = b = c = 0$.

c) If two 3 by 3 matrices A and B satisfy $AB = -BA$, then at least one of A or B is not invertible.