MAT 1341C, INTRODUCTION TO LINEAR ALGEBRA, WINTER 2003

Final Exam, April 30, 2003

Prof. P. Selinger

FAMILY NAME: ________________ **FIRST NAME:** ________________ **ID:** ________________

Question:	1–10	11	12	13	14	15	16	Total
Possible Points	40	12	8	10	10	10	10	100
Actual Points:								

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. You have 3 hours to complete this exam.
2. This is a closed book exam. No notes or calculators are permitted.
3. The exam has 16 questions. Questions 1–10 are multiple choice. Questions 11–12 are short answer questions. Questions 13–16 require a detailed answer. Please write legibly and reason carefully. You can write on the back of pages if necessary. If you need scrap paper, please let a proctor know.
4. Record the LETTERS corresponding to your answer of the multiple choice questions in the boxes below. Do not use pencil.

ANSWERS TO MULTIPLE CHOICE QUESTIONS:

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.

Question 1. Suppose that A and B are invertible matrices of the same size. Which of the following need not be invertible?

A. $A+B$.

B. ABA^{-1}.

C. B^{-1}.

D. AB.

E. The identity matrix.

F. A^{-1}.

Question 2. Consider the system of linear equations given by

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\ \qquad\qquad x_3 + x_4 + x_5 = 2 \\ \qquad\qquad\qquad\qquad x_5 = 3 \end{cases}$$

Determine how many parameters the solution set depends on.

A. 1 parameters.

B. 2 parameters.

C. 5 parameters.

D. 0 parameters.

E. 3 parameters.

F. There is no solution.

Question 3. Find a complex number z such that $\begin{vmatrix} 1 & 1-i & -i \\ 0 & z & 1+i \\ 0 & 0 & 1+i \end{vmatrix} = 3+i$.

A. $z = -i$

B. $z = 2i$

C. $z = 2 - i$

D. $z = 1 - i$

E. $z = 2 + 2i$

F. $z = 1 + 2i$

Question 4. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{pmatrix}$. Find A^{-1} and give its first row.

A. $(1, 0, -2)$.

B. $(-1, 2, -1)$.

C. $(1, 1, -1)$.

D. $(1, -2, 1)$.

E. $(2, -2, 1)$.

F. $(-1, 0, 2)$.

Question 5. Alice, John and Paul have purchased identical crayons and, while playing, got them all mixed up. Now the children need to divide 22 crayons between themselves. Luckily, Alice remembers that she had 3 more crayons than John did. Also, Paul remembers that he had as many crayons as the other two combined. How many crayons did Alice have?

A. 2

B. 5

C. 7

D. 3

E. 12

F. There is not enough data.

Question 6. Find a matrix A such that

$$\left(2A^T + \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\right)^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and give its first row.

A. $(2, -1)$

B. $(0, 0)$

C. $(-1/2, 1/2)$

D. $(0, 1/2)$

E. $(1/2, 0)$

F. $(1, 1/2)$

Question 7. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear mapping, satisfying

$$T(1,2) = (1,0,1) \quad \text{and} \quad T(2,5) = (0,1,1).$$

Calculate $T(0,1)$.

A. $(0,2,-3)$

B. $(-1,1,2)$

C. $(1,1,0)$

D. $(-2,5,1)$

E. $(1,-1,0)$

F. $(-2,1,-1)$

Question 8. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & t & 0 \\ 1 & 0 & -1 \end{pmatrix}$. Find the set of values of t for which the homogeneous system of linear equations $AX = 0$ has a non-trivial solution.

A. $t = -3$

B. $t \neq 2$

C. $t \neq -3$

D. $t \neq 1$ and $t \neq 3$

E. $t = 1$ or $t = 3$

F. $t = 2$

Question 9. Let A, B, C be square invertible matrices satisfying $AB = B^2C$. Assume that $\det B = 3$ and $\det C = 2$. Find a formula for A and calculate the determinant of A.

A. $A = BC, \quad \det A = 6.$

B. $A = B^3C, \quad \det A = 11.$

C. $A = B^2CB^{-1}, \quad \det A = 6.$

D. $A = B^2CB^{-1}, \quad \det A = 5.$

E. $A = B^3C, \quad \det A = 54.$

F. $A = BC, \quad \det A = 5.$

Question 10. Let $A = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 8 & -5 \\ -3 & 10 & -7 \end{pmatrix}$, $X = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}$, $Y = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. Which of the following statements is true?

A. Y is an eigenvector of A with the eigenvalue 2.

B. Y is an eigenvector of A with the eigenvalue -2.

C. Y is an eigenvector of A with the eigenvalue 3.

D. X is an eigenvector of A with the eigenvalue 3.

E. X is an eigenvector of A with the eigenvalue -2.

F. X is an eigenvector of A with the eigenvalue 2.

Question 11 (12 points). (*Short answer question.*)

(a) Let V be a vector space over the field $\mathbb{R}$ of real numbers. What does it mean that vectors $v_1, \ldots, v_n$ in V are linearly independent? Answer with a complete sentence.

(b) Determine, for every statement below, if it is true or false. You will get 4 marks for three correct answers, 2 marks for two correct answers, and 0 marks otherwise.

Every linearly independent set of vectors in a vector space V forms a basis of V.	☐ true	☐ false
Every subset of a linearly independent set of vectors is itself linearly independent.	☐ true	☐ false
If $v_1, \ldots, v_n$ are linearly independent vectors in a vector space V, then $\dim(V) \geq n$.	☐ true	☐ false

(c) Give a basis of the vector space $\mathbf{P}_2(t)$ of all polynomial functions with real coefficients of degree at most 2.

(d) Recall that $\mathbf{M}_2(\mathbb{R})$ is the vector space of real 2×2-matrices. Give a basis of the subspace $W = \{A \in \mathbf{M}_2(\mathbb{R}) \mid A^T = A\}$.

Question 12 (8 points). (*Short answer question.*) Answer each question within the given space.

(i) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear function given by

$$F(x, y, z) = (x + y, y + 2z, x - 2z).$$

Find a basis for the kernel of F.

(ii) Find a basis for the image of F.

(iii) Let $A = \begin{pmatrix} 2 & 2 & 2 & 0 \\ -1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}$. Find the raw canonical form R of A. Find the rank of A and the dimension of the solution space $V = \{X \mid AX = 0\}$ of the system of homogeneous linear equations $AX = 0$.

Answers:

$R =$ $\boxed{}$ $\quad$ rank(A) $=$ $\boxed{}$ $\quad$ dim(V) $=$ $\boxed{}$

Question 13 (10 points).

(a) Let W be a subset of a vector space V over $\mathbb{R}$. State the three conditions which W must satisfy to be a subspace of V.

(1)

(2)

(3)

(b) Prove that $W = \{A \in \mathbf{M}_2(\mathbb{R}) \mid \det(A) = 0\}$ is not a vector subspace of $\mathbf{M}_2(\mathbb{R})$ by giving a *concrete example*, showing that one of the three conditions above fails.

(c) Prove that $W = \{p \in \mathbf{P}(t) \mid p(1) = 0 \text{ and } p(3) = 0\}$ is a vector subspace of the vector space $\mathbf{P}(t)$ of all polynomials with real coefficients, by verifying that W satisfies all three conditions above.

Question 14 (10 points). Let S be the following basis of $\mathbb{R}^3$:

$$S = \{(1,0,0), \quad (0,1,0), \quad (0,1,1)\}.$$

Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear function given by

$$T(x,y,z) = (x+y, x, z), \quad \text{for all } (x,y,z) \in \mathbb{R}^3.$$

(a) Calculate the coordinates $[v]_S$ of the vector $v = (1,1,1)$ in the basis S.

(b) Calculate the coordinates $[v]_S$ of an arbitrary vector $v = (a,b,c)$ in the basis S.

(c) Find a matrix M representing the linear function T in the basis S, that is, such that for all $v \in \mathbb{R}^3$ one has $[T(v)]_S = M[v]_S$.

Question 15 (10 points). Let

$$u = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}.$$

(a) Determine whether or not u, v and w are linearly independent. If they are linearly dependent, give a nontrivial linear combination of u, v and w which equals 0. In any case, fully justify your answer.

(b) Give a basis of the vector subspace $W = \operatorname{span}\{u, v, w\}$ of $\mathbb{R}^4$, spanned by u, v, w.

(c) Find a vector v in $\mathbb{R}^4$ which does not belong to W.

Question 16 (10 points). (a) Find the characteristic polynomial and all eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(b) The eigenvalues of the matrix

$$B = \begin{pmatrix} -5 & 0 & 6 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{pmatrix}$$

are $\lambda_1 = 1$ and $\lambda_2 = -2$. Find all the eigenvectors of B for each of these two eigenvalues. Find a matrix P such that $P^{-1}BP = D$ is diagonal, and give this diagonal matrix D. (There is no need to compute P^{-1}).