MAT 1341, INTRODUCTION TO LINEAR ALGEBRA, WINTER 2003

Answers to the Midterm, March 7

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Question 1. Under what condition can a point (a, b, c) be written as a linear combination of (1, 2, 0) and (1, 1, 1)?

- $\Box \quad 3a b 2c = 0.$
- $\Box \quad a+b-2c=0.$
- $\boxtimes \quad 2a b c = 0.$
- $\Box \quad 2a b + 2c = 0.$
- $\Box \quad a-b=0.$

$$\Box \quad a - 3b + 2c = 0.$$

Question 2. Suppose that a given matrix A satisfies $A^2 - 2A - I = 0$. Give a formula for A^{-1} :

- $\Box \quad A^{-1} = 2A + I.$
- $\Box \quad A^{-1} = A + I.$
- $\boxtimes \quad A^{-1} = A 2I.$
- $\Box \quad A^{-1} = 2A I.$

$$\Box \quad A^{-1} = A + 2I.$$

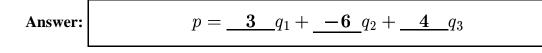
$$\Box \quad A^{-1} = A - I.$$

Question 3. In the vector space $\mathbf{M}_{2,2}$ of real 2×2 -matrices, express M as a linear combination of the matrices A, B, and C, where

$$M = \begin{pmatrix} 4 & 7 \\ 7 & 9 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

Answer:
$$M = \underline{-1}A + \underline{3}B + \underline{2}C$$

Question 4. Write the polynomial $p(t) = t^2 - 2t + 1$ as a linear combination of the three polynomials $q_1(t) = t^2 - 1$, $q_2(t) = t^2 + t$ and $q_3(t) = t^2 + t + 1$.



Question 5. Determine whether or not W is a subspace of \mathbb{R}^3 where W consists of all vectors (a, b, c) in \mathbb{R}^3 such that:

(a)	$b = a^2$	□ yes	🖾 no
(b)	a = 2b = 3c	🖾 yes	🗆 no
(c)	a = 3b	🛛 yes	🗌 no
(d)	ab = 0	□ yes	🛛 no
(e)	$a \leqslant b \leqslant c$	□ yes	🛛 no
(f)	a + b + c = 0	🛛 yes	🗆 no

Question 6. Let V be a vector space over a field K. Which of the following statements are always valid:

(a) Every subset of V is a subspace of V	□ true	⊠ false
(b) Every subspace of V is a subset of V	🖾 true	□ false
(c) $\{0\}$ is a subspace of V	🖾 true	□ false

- (d) Let $u, v \in V$ be vectors, and let W be a subspace of V. If W contains the vectors u and v, then W also contains the sum u + v. \square false
- (e) Let $u, v \in V$ be vectors, and let W be a subspace of V. If W contains the sum u + v, then W also contains u and v. \Box true \boxtimes false

Question 7. Let W be a subset of a vector space V over a field K.

- (a) Write down the three conditions which W must satisfy in order to be a subspace of V:
 - (1) $0 \in W$
 - (2) $\forall v, u(v, u \in W \Rightarrow v + u \in W).$
 - (3) $\forall v, k (v \in W, k \in K \Rightarrow kv \in W).$

(b) Show that $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y \ge z\}$ is not a subspace of \mathbb{R}^3 by showing, in a specific example, that one of the above three conditions is violated.

Answer: The condition (3) is violated. Specific example: v = (1, 1, 1), k = -1. Then $v \in W$, $k \in \mathbb{R}$, but $kv = (-1, -1, -1) \notin W$.

Question 8. (a) Let $V = \mathbf{P}(t)$ be the vector space of polynomials in the variable t, with real coefficients. Let $W = \{p(t) \in V \mid p(2) = p(-2)\}$. Show that W is a subspace of V.

Answer:

- (1) The constant zero polynomial p(t) = 0 is in W, because p(2) = 0 = p(-2).
- (2) Suppose $p(t) \in W$, $q(t) \in W$. Then p(2) = p(-2) and q(2) = q(-2). Let r(t) = p(t)+q(t). Then r(2) = p(2) + q(2) = p(-2) + q(-2) = r(-2), hence $r(t) \in W$.
- (3) Suppose $p(t) \in W$, $k \in \mathbb{R}$. Then p(2) = p(-2). Thus, kp(2) = kp(-2), hence $kp(t) \in W$.

(b) Recall that an $n \times n$ -matrix A is called *symmetric* if $A = A^T$. Prove that the set of all symmetric $n \times n$ -matrices is a subspace of $\mathbf{M}_{n,n}$.

Answer: Let *W* be the subset of symmetric matrices.

- (1) $0 = 0^T$, hence $0 \in W$.
- (2) Suppose $A, B \in W$. Then $A = A^T$ and $B = B^T$. Hence $(A + B)^T = A^T + B^T = A + B$, hence $A + B \in W$.
- (3) Suppose $A \in W$ and $k \in K$. Then $A = A^T$. Thus, $(kA)^T = k(A^T) = kA$, hence $kA \in W$.