MATH/CSCI 2112, DISCRETE STRUCTURES I, FALL 2005

Handout 2: Rules of natural deduction

Conjunction introduction ($\wedge$I)

$$\begin{array}{l|ll} m & A & \\ \vdots & \vdots & \\ n & B & \\ \vdots & \vdots & \\ p & A \wedge B & \wedge\text{I}, m, n \end{array} \qquad \begin{array}{l|ll} m & B & \\ \vdots & \vdots & \\ n & A & \\ \vdots & \vdots & \\ p & A \wedge B & \wedge\text{I}, m, n \end{array}$$

Conjunction elimination ($\wedge$E)

$$\begin{array}{l|ll} m & A \wedge B & \\ \vdots & \vdots & \\ n & A & \wedge\text{E}, m \end{array} \qquad \begin{array}{l|ll} m & A \wedge B & \\ \vdots & \vdots & \\ n & B & \wedge\text{E}, m \end{array}$$

Disjunction Introduction ($\vee$I)

$$\begin{array}{l|ll} m & A & \\ \vdots & \vdots & \\ n & A \vee B & \vee\text{I}, m \end{array} \qquad \begin{array}{l|ll} m & B & \\ \vdots & \vdots & \\ n & A \vee B & \vee\text{I}, m \end{array}$$

Disjunction Elimination ($\vee$E)

$$\begin{array}{l|ll} m & A \vee B & \\ \vdots & \vdots & \\ n & A \rightarrow \varphi & \\ \vdots & \vdots & \\ p & B \rightarrow \varphi & \\ \vdots & \vdots & \\ q & \varphi & \vee\text{E}, m, n, p \end{array}$$

Implication Introduction ($\rightarrow$I)

$$\begin{array}{l|l|l} m & & \underline{A} \\ \vdots & & \vdots \\ n & & B \\ n+1 & \multicolumn{2}{l}{A \rightarrow B \quad \rightarrow\text{I}, m\text{–}n} \end{array}$$

Implication Elimination ($\rightarrow$E)

$$\begin{array}{l|ll} m & A & \\ \vdots & \vdots & \\ n & A \rightarrow B & \\ \vdots & \vdots & \\ p & B & \rightarrow\text{E}, m, n \end{array} \qquad \begin{array}{l|ll} m & A \rightarrow B & \\ \vdots & \vdots & \\ n & A & \\ \vdots & \vdots & \\ p & B & \rightarrow\text{E}, m, n \end{array}$$

Negation Introduction ($\sim$I)

$$\begin{array}{l|l|l} m & & \underline{A} \\ \vdots & & \vdots \\ n & & \bot \\ n+1 & \multicolumn{2}{l}{\sim A \quad \sim\text{I}, m\text{–}n} \end{array}$$

Negation Elimination ($\sim$E)

$$\begin{array}{l|ll} m & A & \\ \vdots & \vdots & \\ n & \sim A & \\ \vdots & \vdots & \\ p & \bot & \sim\text{E}, m, n \end{array} \qquad \begin{array}{l|ll} m & \sim A & \\ \vdots & \vdots & \\ n & A & \\ \vdots & \vdots & \\ p & \bot & \sim\text{E}, m, n \end{array}$$

Contradiction Elimination ($\bot$E)

$$\begin{array}{l|ll} m & \bot & \\ \vdots & \vdots & \\ n & C & \bot\text{E}, m \end{array}$$

Double negation elimination ($\sim\sim$E)

$$\begin{array}{l|ll} m & \sim\sim A & \\ \vdots & \vdots & \\ n & A & \sim\sim\text{E}, m \end{array}$$

Logical equivalence (Eq)

If $A \equiv B$ by logical equivalence (e.g. DeMorgan's law):

$$\begin{array}{l|ll} m & A & \\ \vdots & \vdots & \\ n & B & \text{Eq}, m \end{array}$$

Repetition (R)

$$\begin{array}{l|lll} m & A & & \\ \vdots & \vdots & & \\ \vdots & \cdots & \multicolumn{1}{|l}{\vdots} & \\ n & & \multicolumn{1}{|l}{A} & \text{R}, m \end{array}$$

Forall-introduction ($\forall$I)

$$\begin{array}{l|l|l} m & u & \vdots \\ \vdots & & \vdots \\ n & & A(u) \\ n+1 & \multicolumn{2}{l}{\forall x\, A(x) \quad \forall\text{I}, m\text{–}n} \end{array}$$

Forall-elimination ($\forall$E)

$$\begin{array}{l|ll} m & \forall x\, A(x) & \\ \vdots & \vdots & \\ n & A(t) & \forall\text{E}, m \end{array}$$

Exists-Introduction ($\exists$I)

$$\begin{array}{l|ll} m & A(t) & \\ \vdots & \vdots & \\ n & \exists x\, A(x) & \exists\text{I}, m \end{array}$$

Exists-Elimination ($\exists$E)

$$\begin{array}{l|lll} p & \exists x\, A(x) & & \\ \vdots & \vdots & & \\ m & u & \multicolumn{1}{|l}{\underline{A(u)}} & \\ \vdots & & \multicolumn{1}{|l}{\vdots} & \\ n & & \multicolumn{1}{|l}{\varphi} & \\ n+1 & \varphi & & \exists\text{E}, p, m\text{–}n \end{array}$$

The biconditional ($\leftrightarrow$)

To simplify our formal proof system, we do not introduce any special rules for the connective $\leftrightarrow$. Instead, we simply regard the formula $A \leftrightarrow B$ as an *abbreviation* for $(A \to B) \land (B \to A)$.

Falsity ($\perp$)

The symbol $\perp$ stands for "contradiction" or "falsity". The formula $\perp$ is always false, and it is used in the rules for negation and contradiction above.

Repetition (R)

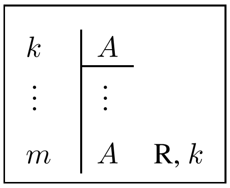
Let A be a formula written at line k (either as a hypothesis, or as a formula already proven). Then one can repeat A at line m if:

(1) $k < m$, and

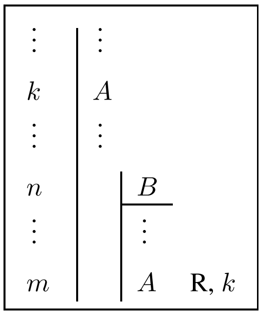
(2) every vertical from line k continues without interruption to line m.

Examples of repetition:

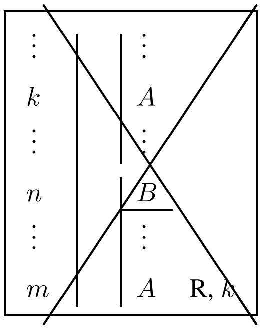
This is ok:



This too:



But not this:



Quantifiers ($\forall$ and $\exists$)

The rules for $\forall$ and $\exists$ are part of predicate logic and will be covered later in the course. They are not used for propositional logic.

Example

Without using the “logical equivalence” rule, we derive one direction of Morgan’s law for disjunction,

$$\sim(A \vee B) \vdash \sim A \wedge \sim B.$$

$$
\begin{array}{r|lll}
1 & \underline{\sim(A \vee B)} & & \\
2 & \quad \vert\ \underline{A} & & \\
3 & \quad \vert\ A \vee B & & \vee\text{I, 2} \\
4 & \quad \vert\ \sim(A \vee B) & & \text{R, 1} \\
5 & \quad \vert\ \bot & & \sim\text{E, 3, 4} \\
6 & \sim A & & \sim\text{I, 2–5} \\
7 & \quad \vert\ \underline{B} & & \\
8 & \quad \vert\ A \vee B & & \vee\text{I, 7} \\
9 & \quad \vert\ \sim(A \vee B) & & \text{R, 1} \\
10 & \quad \vert\ \bot & & \sim\text{E, 8, 9} \\
11 & \sim B & & \sim\text{I, 7–10} \\
12 & \sim A \wedge \sim B & & \wedge\text{I, 6, 11}
\end{array}
$$

Note that there are three other De Morgan’s laws, namely

$$
\begin{array}{rcl}
\sim A \wedge \sim B & \vdash & \sim(A \vee B) \\
\sim(A \wedge B) & \vdash & \sim A \vee \sim B \\
\sim A \vee \sim B & \vdash & \sim(A \wedge B)
\end{array}
$$

Problem 1. Give formal proofs of the remaining three laws of De Morgan.

9. $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.

10. $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$.

11. $A \wedge (A \vee B) \equiv A$.

12. $A \vee (A \wedge B) \equiv A$.

13. $\vdash \neg(A \wedge (\neg A))$.

14. $A \to (\neg A) \vdash \neg A$.

15. $(A \to B) \wedge (A \to C) \equiv A \to (B \wedge C)$.

16. $A \to B,\ B \to C \vdash A \to C$.

17. $\vdash A \to (B \to A)$.

18. $(A \to B) \to C \vdash B \to C$.

19. $A \to B \vdash (\neg B) \to (\neg A)$.

20. $A \to B \vdash \neg(A \wedge (\neg B))$.

21. $A \wedge (\neg B) \vdash \neg(A \wedge B)$.

22. $(\neg A) \to (\neg B) \vdash B \to (\neg(A \to (\neg B)))$.

23. $\vdash \neg(A \leftrightarrow (\neg A))$.

24. $A \vee B \vdash (B \to A) \to A$.

25. $A \vee B \vdash (\neg B) \to (C \to A)$.

26. $(B \to A) \wedge (A \vee B) \vdash A$.

27. $A \vee B \vdash (\neg A) \to B$.

28. $(\neg A) \vee B \vdash A \to B$.

29. $(\neg A) \vee (\neg B) \vdash \neg(A \wedge B)$.

30. $\neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$.

31. $(A \vee B) \to C \equiv (A \to C) \wedge (B \to C)$.

32. $(A \to B) \vee (A \to C) \vdash A \to (B \vee C)$.

33. $(A \to C) \vee (B \to C) \vdash (A \wedge B) \to C$.

34. $((\neg A) \vee C) \wedge (B \to C) \vdash (A \to B) \to C$.

35. $\vdash (A \to C) \to ((B \to C) \to ((A \vee B) \to C))$.

36. $(A \to C) \vee (B \to D) \vdash (A \wedge B) \to (C \vee D)$.

37. $A \to B,\ C \to D,\ (\neg B) \vee (\neg D) \vdash (\neg A) \vee (\neg C)$.

38. $A \to (C \vee D),\ (A \vee D) \vee E,\ \ A \to (\neg C) \vdash D \vee E$.

39. $A \vdash \neg\neg A$.

40. $\neg\neg\neg A \equiv \neg A$.

(NOTE: The $\neg\neg$ rule is required for nos. 41-53.)

41*. $\vdash A \vee (\neg A)$.

42. $(A \to B) \vee C,\ A \to (\neg C) \vdash (B \to C) \to (\neg A)$.

43. $\neg(A \wedge (\neg B)) \vdash A \to B$.

44. $(\neg B) \to (\neg A) \vdash A \to B$.

45*. $A \to B \vdash (\neg A) \vee B$.

46*. $(B \to A) \to A \vdash A \vee B$.

47*. $(A \to B) \to C \vdash A \vee C$.

48*. $(\neg A) \to B \vdash A \vee ((\neg A) \wedge B)$.

49*. $\vdash (A \wedge B) \vee (\neg A) \vee (\neg B)$.

50*. $\neg(A \wedge B) \vdash (\neg A) \vee (\neg B)$.

51*. $\vdash (A \to B) \vee (B \to A)$

52*. $A \to (B \vee C) \vdash (A \to B) \vee (A \to C)$.

53*. $(A \wedge B) \to (C \vee D) \vdash (A \to C) \vee (B \to D)$. (cf.no.36)

54. Prove that if $\varphi \wedge \psi \vdash \theta$ and $\varphi \wedge \theta \vdash \zeta$ then $\varphi \wedge \psi \vdash \zeta$.

55. Prove that if $\varphi \vdash \psi$ then

(i) $\varphi \wedge \theta \vdash \psi \wedge \theta$,
(ii) $\varphi \vee \theta \vdash \psi \vee \theta$,
(iii) $\theta \to \varphi \vdash \theta \to \psi$,
(iv) $\psi \to \theta \vdash \varphi \to \theta$.

56. Prove that $\Gamma \cup \{\varphi\} \vdash \psi$ iff $\Gamma \vdash \varphi \to \psi$ (Γ is any set of formulae).