MAT 3321, COMPLEX ANALYSIS AND INTEGRAL TRANSFORMS, WINTER 2005

Answers to Homework 5
13.1 #4,14,18,20,22,26

Problem 13.1 #4 Recall that the equation

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$$

defines an ellipse with x-intercepts $\pm a$ and y-intercepts $\pm b$. The equation $4x^2 + 9y^2 = 36$ can be written in this form, with $a = 3$ and $b = 2$. It is therefore the equation of an ellipse.

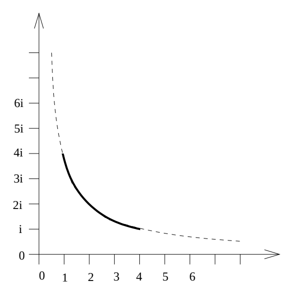
A parametric interpretation of the ellipse with x-intercepts $\pm a$ and y-intercepts $\pm b$ is

$$x(t) = a\cos t, y(t) = b\sin t, \text{ where } t = 0, \ldots, 2\pi.$$

With $a = 3$, $b = 2$, and $z(t) = x(t) + iy(t)$, we therefore get

$$z(t) = 3\cos t + 2i\sin t,\ (t = 0, \ldots, 2\pi).$$

Problem 13.1 #14 The curve $z(t) = t + 4i/t$ can be separated into x- and y-coordinates: $x = t$, $y = 4/t$. Eliminating t from the equation, we get $y = 4/x$, which is the equation of a hyperbola. Further, we have $1 \leqslant t \leqslant 4$, therefore $1 \leqslant x \leqslant 4$. The corresponding piece of hyperbola is sketched as a solid curve in the illustration.



Problem 13.1 #18 We have $f(z) = \bar{z}$. The path C of integration is the parabola $y = x^2$ from 0 to $1 + i$, in other words, from $x = 0$ to $x = 1$. We can easily parameterize this curve by setting $x = t$, $y = t^2$, $t = 0, \ldots, 1$. Thus, $z(t) = t + it^2$. Therefore, the integral in question is

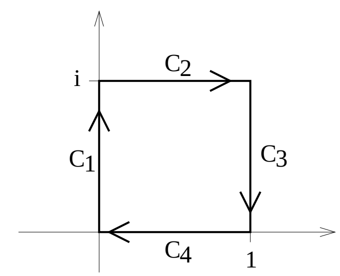
$$\int_C f(z)\ dz = \int_0^1 f(z(t))\dot{z}(t)dt = \int_0^1 \overline{(t + it^2)}(1 + 2it)dt$$

$$= \int_0^1 (t - it^2)(1 + 2it)dt = \int_0^1 t - it^2 + 2it^2 + 2t^3 dt$$

$$= \left[t^2/2 - i/3t^3 + 2i/3t^3 + 2/4t^4\right]_0^1$$

$$= 1/2 - i/3 + 2i/3 + 1/2 = 1 + i/3.$$

Problem 13.1 #20 We have $f(z) = \operatorname{Re} z^2 = x^2 - y^2$. C is the boundary of a square with vertices $0, i, 1+i, 1$, clockwise. The path C can be written as $C = C_1 + C_2 + C_3 + C_4$ for the four paths shown in the illustration. $C_1, \ldots, C_4$ can be parameterized individually:



$C_1 : z(t) = ti\ (t = 0, \ldots, 1)$
$C_2 : z(t) = t + i\ (t = 0, \ldots, 1)$
$C_3 : z(t) = 1 + i - ti\ (t = 0, \ldots, 1)$
$C_4 : z(t) = 1 - t\ (t = 0, \ldots, 1)$

We have:

$$\int_{C_1} f(z)dz = \int_0^1 (0^2 - t^2)idt = -i/3$$
$$\int_{C_2} f(z)dz = \int_0^1 (t^2 - 1^2)1dt = -2/3$$
$$\int_{C_3} f(z)dz = \int_0^1 (1^2 - (1-t)^2)(-i)dt = -2i/3$$
$$\int_{C_4} f(z)dz = \int_0^1 ((1-t)^2 - 0^2)(-1)dt = -1/3$$

Therefore

$$\int_C f(z)dz = -i/3 - 2/3 - 2i/3 - 1/3 = -1 - i.$$

Problem 13.1 #22 Here, $f(z) = \sinh \pi z$, and $z(t) = i(1-t)$ $(t = 0, \ldots, 1)$. Thus

$$\int_C f(z)dz = \int_0^1 f(z(t))\dot{z}(t)dt = \int_0^1 \sinh(\pi i(1-t))(-i)dt$$

$$= \left[\frac{1}{\pi}\cosh(\pi i(1-t))\right]_0^1 = \frac{1}{\pi}\cosh(0) - \frac{1}{\pi}\cosh(\pi i) = \frac{1}{\pi}(1 - (-1)) = \frac{2}{\pi}$$

Problem 13.1 #26 Here we have

$$f(z) = \frac{3}{z-i} - \frac{6}{(z-i)^2}$$

and the path of integration is the circle $|z-i| = 5$, clockwise. This circle can be parameterized as

$$z(t) = i + 5e^{-it}, (t = 0, \ldots, 2\pi).$$

Note that we have used e^{-it}, not e^{it}, to account for the clockwise movement. Thus

$$\int_C f(z)dz = \int_0^{2\pi} f(z(t))\dot{z}(t)dt = \int_0^{2\pi} \left(\frac{3}{5e^{-it}} - \frac{6}{(5e^{-it})^2} \right) (-5ie^{-it})dt$$

$$= \int_0^{2\pi} \left(\frac{-3i}{1} - \frac{-6i}{5e^{-it}} \right) dt = -6\pi i + \frac{6i}{5} \int_0^{2\pi} e^{it} dt = -6\pi i$$