MAT 3361, INTRODUCTION TO MATHEMATICAL LOGIC, Fall 2004

Answers to Problem Set 1: Handout 1, Problems 1-4, 10, 20, 30, 50

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Problem 1 (a) $\neg(A \wedge B) \vdash \neg A \vee \neg B$

1	$\neg(A \wedge B)$	
2	$\quad \neg(\neg A \vee \neg B)$	
3	$\quad\quad A$	
4	$\quad\quad\quad B$	
5	$\quad\quad\quad A \wedge B$	$\wedge$I, 3, 4
6	$\quad\quad\quad \bot$	$\neg$E, 1, 5
7	$\quad\quad \neg B$	$\neg$I, 4–6
8	$\quad\quad \neg A \vee \neg B$	$\vee$I, 7
9	$\quad\quad \bot$	$\neg$E, 2, 8
10	$\quad \neg A$	$\neg$I, 3–9
11	$\quad \neg A \vee \neg B$	$\vee$I, 10
12	$\quad \bot$	$\neg$E, 2, 11
13	$\neg\neg(\neg A \vee \neg B)$	$\neg$I, 2–12
14	$\neg A \vee \neg B$	$\neg\neg$E, 13

(b) $\neg A \wedge \neg B \vdash \neg(A \vee B)$

1	$\neg A \wedge \neg B$	
2	$\quad A \vee B$	
3	$\quad\quad A$	
4	$\quad\quad \neg A$	$\wedge$E, 1
5	$\quad\quad \bot$	$\neg$E, 3, 4
6	$\quad\quad B$	
7	$\quad\quad \neg B$	$\wedge$E, 1
8	$\quad\quad \bot$	$\neg$E, 6, 7
9	$\quad \bot$	$\vee$E, 2, 3–5, 6–8
10	$\neg(A \vee B)$	$\neg$I, 2–9

(c) $\neg A \vee \neg B \vdash \neg(A \wedge B)$

1	$\neg A \vee \neg B$	
2	$\quad A \wedge B$	
3	$\quad\quad \neg A$	
4	$\quad\quad A$	$\wedge$E, 2
5	$\quad\quad \bot$	$\neg$E, 3, 4
6	$\quad\quad \neg B$	
7	$\quad\quad B$	$\wedge$E, 2
8	$\quad\quad \bot$	$\neg$E, 6, 7
9	$\quad \bot$	$\vee$E, 1, 3–5, 6–8
10	$\neg(A \wedge B)$	$\neg$I, 2–9

Problem 2 (a) The definition of rank is:

$$
\begin{aligned}
&r(p_i) = 0\\
&r(\bot) = 0\\
&r(\boldsymbol{(}\varphi \,\Box\, \psi\boldsymbol{)}) = 1 + \max\{r(\varphi), r(\psi)\}\\
&r(\boldsymbol{(}\neg\varphi\boldsymbol{)}) = 1 + r(\varphi).
\end{aligned}
$$

The number of connectives of a formula can be defined as:

$$\begin{aligned}
&c(p_i) = 0\\
&c(\bot) = 1\\
&c((\varphi \,\square\, \psi)) = 1 + c(\varphi) + c(\psi)\\
&c((\neg\varphi)) = 1 + c(\varphi).
\end{aligned}$$

(b) We prove that $r(\varphi) \leqslant c(\varphi)$, for all φ. Base case: $r(p_i) = 0 \leqslant 0 = c(p_i)$ and $r(\bot) = 0 \leqslant 1 = c(\bot)$. Induction step: If $r(\varphi) \leqslant c(\varphi)$ and $r(\psi) \leqslant c(\psi)$, then

$$\begin{aligned}
r((\varphi \,\square\, \psi)) &= 1 + \max\{r(\varphi), r(\psi)\}\\
&\leqslant 1 + r(\varphi) + r(\psi)\\
&\leqslant 1 + c(\varphi) + c(\psi)\\
&= c((\varphi \,\square\, \psi)).
\end{aligned}$$

Here, we used the fact that the maximum of two non-negative numbers is $\leqslant$ their sum. For the remaining induction step, assume $r(\varphi) \leqslant c(\varphi)$. Then

$$\begin{aligned}
r((\neg\varphi)) &= 1 + r(\varphi)\\
&\leqslant 1 + c(\varphi)\\
&= c((\neg\varphi)).
\end{aligned}$$

Problem 3 Let $l(\varphi)$ denote the length of a proposition in symbols. Note that this is always a positive integer. We prove the following statement by induction: "for all propositions φ, $l(\varphi) \notin \{2, 3, 6\}$".

Base case: if φ is atomic, then $l(\varphi) = 1 \notin \{2, 3, 6\}$.

Induction step 1: if $\varphi = (\varphi' \,\square\, \varphi'')$, then $l(\varphi) = 3 + l(\varphi') + l(\varphi'')$. Evidently this quantity is not 2 or 3. Also, if it were 6, then $l(\varphi') + l(\varphi'') = 3$, but this is not possible since then either $l(\varphi') = 2$ or $l(\varphi'') = 2$, contradicting the induction hypothesis.

Induction step 2: if $\varphi = (\neg\varphi')$, then $l(\varphi) = 3 + l(\varphi')$. Evidently this quantity is not 2 or 3; also it cannot be 6 since by induction hypothesis, $l(\varphi') \neq 3$.

Problem 4 (a) Base case: if $\varphi = p_i$ is atomic, then $\varphi = dm(\varphi)$, hence $r(\varphi) = r(dm(\varphi))$. Induction step: assume $\varphi = (\varphi' \wedge \varphi'')$. By induction hypothesis, $r(\varphi') = r(dm(\varphi'))$ and $r(\varphi'') = r(dm(\varphi''))$. But then

$$\begin{aligned}
r(\varphi) &= \;= 1 + max\{r(\varphi'), r(\varphi'')\}\\
&= 1 + max\{r(dm(\varphi')), r(dm(\varphi''))\}\\
&= r((dm(\varphi') \vee dm(\varphi'')))\\
&= r(dm(\varphi)).
\end{aligned}$$

The case for $\varphi = (\varphi' \vee \varphi'')$ is similar. Finally, assume $\varphi = (\neg\varphi')$. By induction hypothesis, $r(\varphi') = r(dm(\varphi'))$. But then

$$\begin{aligned}
r(\varphi) &= \;= 1 + r(\varphi')\\
&= 1 + r(dm(\varphi'))\\
&= r((\neg dm(\varphi')))\\
&= r(dm(\varphi)).
\end{aligned}$$

(b) We have:

$$\begin{aligned}
[\![\varphi \wedge \psi]\!]' &= 1 - [\![dm(\varphi \wedge \psi)]\!]\\
&= 1 - [\![dm(\varphi) \vee dm(\psi)]\!]\\
&= 1 - \max\{[\![dm(\varphi)]\!], [\![dm(\psi)]\!]\}\\
&= \min\{1 - [\![dm(\varphi)]\!], 1 - [\![dm(\psi)]\!]\}\\
&= \min\{[\![\varphi]\!]', [\![\psi]\!]'\}
\end{aligned}$$

$$\begin{aligned}
[\![\varphi \vee \psi]\!]' &= 1 - [\![dm(\varphi \vee \psi)]\!]\\
&= 1 - [\![dm(\varphi) \wedge dm(\psi)]\!]\\
&= 1 - \min\{[\![dm(\varphi)]\!], [\![dm(\psi)]\!]\}\\
&= \max\{1 - [\![dm(\varphi)]\!], 1 - [\![dm(\psi)]\!]\}\\
&= \max\{[\![\varphi]\!]', [\![\psi]\!]'\}
\end{aligned}$$

$$\begin{aligned}
[\![\neg\varphi]\!]' &= 1 - [\![dm(\neg\varphi)]\!]\\
&= 1 - [\![\neg dm(\varphi)]\!]\\
&= 1 - (1 - [\![dm(\varphi)]\!])\\
&= 1 - [\![\varphi]\!]'
\end{aligned}$$

Thus, $[\![-]\!]'$ is a valuation.

(c) In this proof, we use the fact that $dm(dm(\varphi)) = \varphi$, which is easily shown by induction.

Suppose φ is satisfiable. Then there is some valuation $[\![-]\!]$ such that $[\![\varphi]\!] = 1$. Then $[\![-]\!]'$ is a valuation, and we have $[\![dm(\varphi)]\!]' = 1 - [\![dm(dm(\varphi))]\!] = 1 - [\![\varphi]\!] = 0$, thus $dm(\varphi)$ is not valid.

Conversely, suppose that $dm(\varphi)$ is not valid, then there exists some valuation $[\![-]\!]$ such that $[\![dm(\varphi)]\!] = 0$. Then $[\![-]\!]'$ is a valuation, and we have $[\![\varphi]\!]' = 1 - [\![dm(\varphi)]\!] = 1$, hence φ is satisfiable.

Problem 20 $A \Rightarrow B \vdash \neg(A \wedge \neg B)$.

1	$A \Rightarrow B$	
2	$A \wedge \neg B$	
3	A	$\wedge$E, 2
4	$A \Rightarrow B$	R, 1
5	B	$\Rightarrow$E, 3, 4
6	$\neg B$	$\wedge$E, 2
7	$\perp$	$\neg$E, 5, 6
8	$\neg(A \wedge \neg B)$	$\neg$I, 2–7

Problem 10 (a) $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

1	$A \vee (B \wedge C)$	
2	A	
3	$A \vee B$	$\vee$I, 2
4	$A \vee C$	$\vee$I, 2
5	$(A \vee B) \wedge (A \vee C)$	$\wedge$I, 3, 4
6	$B \wedge C$	
7	B	$\wedge$E, 6
8	C	$\wedge$E, 6
9	$A \vee B$	$\vee$I, 7
10	$A \vee C$	$\vee$I, 8
11	$(A \vee B) \wedge (A \vee C)$	$\wedge$I, 9, 10
12	$(A \vee B) \wedge (A \vee C)$	$\vee$E, 2, 3–5, 6–11

(b) $(A \vee B) \wedge (A \vee C) \vdash A \vee (B \wedge C)$

1	$(A \vee B) \wedge (A \vee C)$	
2	$A \vee B$	$\wedge$E, 1
3	$A \vee C$	$\wedge$E, 1
4	A	
5	$A \vee (B \wedge C)$	$\vee$I, 4
6	B	
7	A	
8	$A \vee (B \wedge C)$	$\vee$I, 4
9	C	
10	$B \wedge C$	$\wedge$I, 6, 9
11	$A \vee (B \wedge C)$	$\vee$I, 10
12	$A \vee (B \wedge C)$	$\vee$E, 3, 7–8, 9–11
13	$A \vee (B \wedge C)$	$\vee$E, 2, 4–5, 6–12

Problem 30 See Problem 1.

Problem 50 See Problem 1.