MATH 4135/5135: INTRO. TO CATEGORY THEORY, FALL 2005
Course Notes
Peter Selinger

1 Categories, functors, natural transformations

1.1 Lecture 1, Sep 13, 2005

- **Definition of a category.** A category $\mathbf{C}$ consists of
  - a class $|\mathbf{C}|$ of *objects* $A, B, \ldots$,
  - a set $\hom_{\mathbf{C}}(A, B)$ of *morphisms* (or *arrows*) for any pair of objects A, B (we write $f : A \to B$ if $f \in \hom_{\mathbf{C}}(A, B)$),
  - an *identity* morphism $\mathrm{id}_A : A \to A$ for each object A,
  - for all $f : A \to B$, $g : B \to C$, there is a *composition* morphism $g \circ f : A \to C$,
  - subject to the equations, for all A, B, C, D and $f : A \to B$, $g : B \to C$, $h : C \to D$:

$$\mathrm{id}_B \circ f = f = f \circ id_A \qquad (h \circ g) \circ f = h \circ (g \circ f).$$

- **Class vs. set.** We want to be able to consider the "category of all sets". We therefore requires the objects to form a *class*, rather than a *set*. This is a formality intended to avoid set-theoretic paradoxes (we are not allowed to form the "set of all sets"). If some category indeed has a *set* of objects, we also call it a *small category*. For the most part, we will ignore such cardinality issues, unless a particular situation requires special care.

- **Examples of categories.**
  - **Set** (sets and functions)
  - **Grp** (groups and group homomorphisms)
  - **Ab** (abelian groups and group homomorphisms)
  - **Rng** (rings and ring homomorphisms)
  - **Top** (topological spaces and continuous functions)
  - $\mathbf{Vec}_k$ (vector spaces over field k and linear functions)
  - **Rel** (sets and relations)

  - etc.

- **Examples of categorical definitions.**
  - morphisms $f : A \to B$ and $g : B \to A$ are *inverses* of each other if $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. A morphism f is *invertible* if it has an inverse, in which case the inverse is necessarily unique (Proof: suppose $g, h : B \to A$ are two inverses of f, then

$$g = g \circ \mathrm{id}_b = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{id}_A \circ h = h,$$

    so $g = h$.) We write $f^{-1} : B \to A$ for the unique inverse of $f : A \to B$, if any.
  - a morphism $f : A \to B$ is called a *monomorphism* or *monic* if for all objects C and all $g, h : C \to A$, $f \circ g = f \circ h$ implies $g = h$. Lemma: in the category of sets and functions, the monomorphisms are precisely the injective functions.

1.2 Lecture 2, Sep 15, 2005

- **Notations.** We use the following notations in a category $\mathbf{C}$:
  - The hom-set $\hom_{\mathbf{C}}(A, B)$ is also written $\mathbf{C}(A, B)$, or if the category $\mathbf{C}$ is clear from context we also write $\hom(A, B)$ or simply (A, B).
  - If $f \in (A, B)$, we write $f : A \to B$ or $A \xrightarrow{f} B$. Moreover, A is called the *domain* of f and B is called the *codomain* of f. We write $\mathrm{dom}(f) = A$, $\mathrm{cod}(f) = B$.
  - Diagrams: a diagram such as

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

    is a notation for an *equation* such as $g \circ f = k \circ h$. In particular, when we write a diagram, we always mean that it commutes. If a part of a diagram is *not* assumed to commute, we indicate this explicitly by the symbol “⊹”, as in:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "⊹"] & B \arrow[d, "g"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

Note that the symbol “⊹” only removes *one* equation, so for example, the diagram

$$\begin{tikzcd} C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[r, phantom, "⊹"] & A \arrow[r, "f"] & B \end{tikzcd}$$

means $g \circ f = h \circ f$ (but not necessarily $g = h$).

- **Graphs.** We briefly discuss an alternate presentation of the definition of a category in terms of *graphs*. A *graph* consists of
  - a class $\mathcal{V}$ of *vertices* $A, B, \ldots$,
  - a class $\mathcal{A}$ of *arrows* $f, g, \ldots$,
  - two operations dom : $\mathcal{V} \to \mathcal{A}$ and cod : $\mathcal{V} \to \mathcal{A}$, called the *domain* and *codomain* operation.

  Finite graphs can be visualized:

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[loop right] \arrow[dl, shift left] \\ \bullet \arrow[u] \arrow[ur, shift left] & \end{tikzcd}$$

  Note that this notion of graph allows *multiple arrows* between two vertices, and it also allows arrows from one vertex to itself (loops).

- In a graph, we write $(A, B) = \{f \mid \mathrm{dom}(f) = A, \mathrm{cod}(f) = B\}$.

- A *category* is a graph with additional constants and operations

$$\mathrm{id}_A \in (A, A) \qquad \circ : (A, B) \times (B, C) \to (A, C)$$

  satisfying the associativity and unit laws $(h \circ g) \circ f = h \circ (g \circ f)$, $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$, whenever f, g, h are of the appropriate types. (In addition, we also require (A, B) to be a *set* for all A, B, a property sometimes called *local smallness*. Not everybody requires this).

- The definition of categories in terms of graphs reveals the combinatorial character of the definition, and it allows us to visualize finite categories by drawing their graphs (and discussing the composition operation).
  - **0** is the unique category with 0 objects, also known as the *empty category*.
  - **1** is the unique category with one object and one morphism. Its underlying graph is:

$$\begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd}$$

  - **2** is the following category with two objects and three morphisms:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[loop below] & \bullet \arrow[loop below] \end{tikzcd}$$

    There is a unique composition operation on this graph.
  - **3** is the following category with three objects and six morphisms:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend left] \arrow[loop below] & \bullet \arrow[r] \arrow[loop below] & \bullet \arrow[loop below] \end{tikzcd}$$

    We often omit the identity arrows when we picture a category as a graph, yielding the simpler picture:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend left] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

  - $\downdownarrows$ is the unique category with the following underlying graph (identity arrows not pictured):

$$\bullet \rightrightarrows \bullet$$

- **Concrete and abstract categories.** Categories where the objects are *spaces* and the morphisms are *structure-preserving functions*, such as **Set**, **Grp**, and most of the examples from Lecture 1, are also known as *concrete categories*. All other categories, such as **0**, **1**, **2**, $\downdownarrows$, are known as *abstract categories*.

- **Monoids.** A category with one object is called a *monoid*. Equivalently, a monoid can be described as a set M together with a multiplication operation $\cdot : M \times M \to M$ and a unit element $e \in M$, such that for all $x, y, z \in M$,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \qquad e \cdot x = x = x \cdot e.$$

(In other words, like a group, but without inverses). Note that the elements of M form the morphisms of a one-object category (where the unique object does not need a name).

If $(M, \cdot, e)$ and $(N, \cdot, e')$ are monoids, then a *monoid homomorphism* is function $f : M \to N$ such that $f(x \cdot y) = f(x) \cdot f(y)$ and $f(e) = e'$, for all $x, y \in M$.

We have seen that each monoid can be regarded as a one-object (abstract) category.

The class of *all* monoids, together with monoid homomorphisms, also forms a (concrete) category, which we call **Mon**.

- **Groups.** A *group* can be described as a one-object category in which every morphism is invertible.

- **Preorders.** A category where there is at most one arrow between any pair of objects A, B is called a *preorder*. Equivalently, a preorder can be described as a set (or class) P, together with a binary relation $\leqslant$, subject to these axioms, for all $x, y, z \in P$:

$$\begin{array}{ll} x \leqslant x & \text{(reflexivity)} \\ x \leqslant y, y \leqslant z \Rightarrow x \leqslant z & \text{(transitivity)} \end{array}$$

Here, the elements of P form the *objects* of a category, and there is a unique morphism $f : x \to y$ iff $x \leqslant y$.

Given two preorders P, Q, a function $f : P \to Q$ is called *monotone* if for all $x, y \in P$, $x \leqslant y \Rightarrow f(x) \leqslant f(y)$.

We have seen that each preorder can be regarded as an (abstract) category.

The class of *all* preorders, together with monotone functions, also forms a (concrete) category, which we call **Pre**.

- **Partial orders, linear orders.** A preorder $(P, \leqslant)$ is called a *partial order* if it satisfies the axiom of antisymmetry. A partial order is moreover called a *total order* if it also satisfies the axioms of totality.

$$\begin{array}{ll} x \leqslant y \wedge y \leqslant x \Rightarrow x = y & \text{(antisymmetry)} \\ \forall x, y \in P (x \leqslant y \vee y \leqslant x) & \text{(totality)} \end{array}$$

A partially ordered set is also called a *poset*. The category of posets and monotone maps is called **Poset**.

1.3 Lecture 3, Sep 20, 2005

- **Discrete categories.** A category is *discrete* if it has no morphisms except identity morphisms. A discrete category is essentially uniquely determined by its class of objects. We sometimes identify a discrete category with its class of objects, so for example, if $\mathbf{C}$ is a category, we write $|\mathbf{C}|$ to denote the class of objects of $\mathbf{C}$, but we also write $|\mathbf{C}|$ to denote the discrete category that has the same objects as $\mathbf{C}$.

- **Duality.** If $\mathbf{C}$ is a category, we write $\mathbf{C}^{op}$ for the category obtained from $\mathbf{C}$ by reversing the direction of all the arrows. $\mathbf{C}^{op}$ is called the *dual* or *opposite* category of $\mathbf{C}$. Formally, we have: $|\mathbf{C}^{op}| = |\mathbf{C}|$, $\mathbf{C}^{op}(A, B) = \mathbf{C}(B, A)$, and $g \circ_{\mathbf{C}^{op}} f = f \circ_{\mathbf{C}} g$.

- **Dual of a property or construction.** Because each category has a dual category, each property of categories also has a dual property. For example, $f : A \to B$ is called an *epimorphism* in $\mathbf{C}$ if $f : B \to A$ is a *monomorphism* in $\mathbf{C}^{op}$. Concretely, $f : A \to B$ is an epimorphism if for all objects C, and all pairs of morphism $g, h : B \to C$, $g \circ f = h \circ f$ implies $g = h$. Compare this to the definition of a monomorphism from Lecture 1. In the category **Set**, the epimorphisms are exactly the surjective functions.

- **Functors.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* $F : \mathbf{C} \to \mathbf{D}$ is given by the following data:

  - an object map $F : |\mathbf{C}| \to |\mathbf{D}|$,
  - for any two objects $A, B \in \mathbf{C}$, a morphism map $F : \mathbf{C}(A, B) \to \mathbf{D}(FA, FB)$,
  - such that $F(\mathrm{id}_A) = \mathrm{id}_{FA}$ and $F(g \circ f) = Fg \circ Ff$, for all $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$.

  In other words, a functor maps each morphism $A \xrightarrow{f} B$ in $\mathbf{C}$ to a morphism $FA \xrightarrow{Ff} FB$ in $\mathbf{D}$, and preserves identities and composition.

- **Examples of functors on Set.**

  - **The covariant powerset functor.** The *powerset* of a set X is

$$\mathscr{P}X = \{U \mid U \subseteq X\}.$$

Given a function $f : X \to Y$, we can define $\mathscr{P}f : \mathscr{P}X \to \mathscr{P}Y$ to be the *direct image* operation: for all $U \in \mathscr{P}X$, define

$$(\mathscr{P}f)(U) = f[U] = \{f(x) \mid x \in U\}.$$

With this assignment, the powerset operation is a functor $\mathscr{P}$: **Set** $\to$ **Set**. Indeed, we check that identities and composition are preserved:

$$\begin{array}{lcl}
(\mathscr{P}\,\mathrm{id}_X)(U) & = & \{\mathrm{id}_X(x) \mid x \in U\} \\
 & = & \{x \mid x \in U\} \\
 & = & U \\
 & = & \mathrm{id}_{\mathscr{P}X}(U), \\
(\mathscr{P}g \circ \mathscr{P}f)(U) & = & (\mathscr{P}g)(\{f(x) \mid x \in U\}) \\
 & = & \{g(y) \mid y \in \{f(x) \mid x \in U\}\} \\
 & = & \{g(f(x)) \mid x \in U\} \\
 & = & \{g \circ f(x) \mid x \in U\} \\
 & = & (\mathscr{P}(g \circ f))(U).
\end{array}$$

– **Multiplication by a fixed set.** Let A be a fixed set, and consider the operation $FX = X \times A$ on sets, where "$\times$" denotes the usual cartesian product of sets. To make this into a functor F : **Set** $\to$ **Set**, we need to define, for any function $f : X \to Y$, a function $Ff : FX \to FY$, i.e., $Ff : X \times A \to Y \times A$. There is only one obvious way to do so, namely:

$$Ff(x, a) = (fx, a).$$

One easily checks that $F(g \circ f) = Fg \circ Ff$ and $F\,\mathrm{id} = \mathrm{id}$, so this is indeed a functor.

Note: it is common to denote the action of a functor on morphism by the same symbols as the action on objects. We therefore also write the function Ff as

$$X \times A \xrightarrow{f \times A} Y \times A.$$

– **Disjoint union with a fixed set.** As in the previous example, let A be a fixed set, and define $FX = X + A$, where "$+$" denotes disjoint union of sets. This can be made into a functor F : **Set** $\to$ **Set** by defining, for each $f : X \to Y$, a function $Ff : X + A \to Y + A$ in the obvious way.

– **The finite powerset functor.** If X is a set, let us write $\mathscr{P}_{\mathrm{fin}}X = \{U \subseteq X \mid U \text{ is finite}\}$. $\mathscr{P}_{\mathrm{fin}}X$ is called the *finite powerset* of X. The $\mathscr{P}_{\mathrm{fin}}$ operation can be made into a functor $\mathscr{P}_{\mathrm{fin}}$: **Set** $\to$ **Set** in the same way as $\mathscr{P}$.

- **Examples of functors between two different categories.**

  – To each group G, we can associate its underlying set of elements $|G|$. To each group homomorphism $f : G \to H$, we can associate its underlying function $g : |G| \to |H|$. This information defines a functor F : **Grp** $\to$ **Set**, namely:

$$\begin{array}{l} F(G) = |G|, \\ F(f) = f \end{array}$$

  This functor is called the *forgetful functor* from **Grp** to **Set**, because it does nothing except forget part of the structure of G and part of the properties of f.

  – Similarly, there are obvious forgetful functors F : **Top** $\to$ **Set** (mapping each topological space to its underlying set), F : **Rng** $\to$ **Ab** (mapping each ring $(R, +, \cdot)$ to its underlying abelian group $(R, +)$), etc.

- **Covariant and contravariant functors.** A *contravariant functor* is like a functor, except that it reverses the direction of the arrows. It maps objects A to FA, and morphisms $A \xrightarrow{f} B$ to $FB \xrightarrow{Ff} FA$. It preserves identities in the usual way, and composition in the reverse sense, i.e., $F(g \circ f) = Ff \circ Fg$.

  Equivalently, a contravariant functor F from $\mathbf{C}$ to $\mathbf{D}$ is simply an ordinary functor $F : \mathbf{C}^{op} \to \mathbf{D}$.

  A functor that is not contravariant (i.e., an ordinary functor) is also sometimes called a *covariant* functor.

- **Examples of contravariant functors.**

  – **The contravariant powerset functor.** The powerset operation can also be given the structure of a contravariant functor. Namely, given $f : X \to Y$, we can define the *inverse image* function $f^{-1} : \mathscr{P}Y \to \mathscr{P}X$ via $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$, for $V \in \mathscr{P}Y$. The the assignment $FX = \mathscr{P}X$, $Ff = f^{-1}$ defines a contravariant functor **Set** $\to$ **Set**.

- **The exponential functor.** Let A be a fixed set. Let us write $A^X = \{f \mid f : X \to A\}$ for the set of all functions from X to A. The operation $FX = A^X$ extends naturally to a contravariant functor. Namely, given $f : X \to Y$, we can define $A^f : A^Y \to A^X$ as follows:

$$A^f(s) = s \circ f,$$

  where $s \in A^Y$ is a function $s : Y \to A$.

- **Functors of more than one variable.** Let $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ be categories. A *functor in two variables*, also known as a *bifunctor*, from $\mathbf{C}$ and $\mathbf{D}$ to $\mathbf{E}$, is given by the following:

  - to each pair of objects $A \in |\mathbf{C}|$ and $B \in |\mathbf{D}|$, we associate an object $F(A, B) \in |\mathbf{E}|$.
  - to each pair of morphisms $f : A \to A'$ in $\mathbf{C}$ and $g : B \to B'$ in $\mathbf{D}$, we associate a morphism $F(A, B) \xrightarrow{F(f,g)} F(A', B')$ in $\mathbf{E}$,
  - identities and composition are preserved simultaneously, i.e.,

$$\begin{array}{l} F(\mathrm{id}_A, \mathrm{id}_B) = \mathrm{id}_{F(A,B)} : F(A, B) \to F(A, B) \\ F(f' \circ f, g' \circ g) = F(f', g') \circ F(f, g), \end{array}$$

  for all $f : A \to A'$, $f' : A' \to A''$ in $\mathbf{C}$ and $g : B \to B'$, $g' : B' \to B''$ in $\mathbf{D}$.

  The composition property can be written as a diagram:

$$\begin{tikzcd} F(A,B) \arrow[r, "F(f,g)"] \arrow[dr, "F(f' \circ f, g' \circ g)"'] & F(A',B') \arrow[d, "F(f',g')"] \\ & F(A'',B'') \end{tikzcd}$$

  Functors of 3, 4, or more variables are defined analogously.

- **Example of a bifunctor.** $F(X, Y) = X \times Y$ defines a bifunctor in the category of sets, where $F(f, g)(x, y) = (f(x), g(y))$.

- **The cartesian product of categories.** Let $\mathbf{C}, \mathbf{D}$ be two categories. The *cartesian product* of $\mathbf{C}$ and $\mathbf{D}$ is the category $\mathbf{E}$ whose objects are pairs of objects, and whose morphisms are pairs of morphisms of $\mathbf{C}$ and $\mathbf{D}$. More

precisely, we have $|\mathbf{E}| = |\mathbf{C}| \times |\mathbf{D}|$, and $\mathbf{E}((A, B), (A', B')) = \mathbf{C}(A, A') \times \mathbf{D}(B, B')$. Concretely, this means that if $f : A \to A'$ in $\mathbf{C}$ and $g : B \to B'$ in $\mathbf{D}$, then

$$(f, g) : (A, B) \to (A', B')$$

in $\mathbf{E}$. Identities and composition are componentwise, so e.g.

$$(f', g') \circ (f, g) = (f' \circ f, g' \circ g).$$

We write $\mathbf{E} = \mathbf{C} \times \mathbf{D}$.

- **Bifunctors as ordinary functors.** A bifunctor F from $\mathbf{C}$ and $\mathbf{D}$ to $\mathbf{E}$ can be equivalently described as an (ordinary) functor $F : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$, where $\mathbf{C} \times \mathbf{D}$ is the cartesian product of categories.

- **Mixed variance functors.** It is even possible to have a functor in several variables that is covariant in some variables and contravariant in others. Such a functor is said to have *mixed variance*. For example, consider a functor F that maps $X \in \mathbf{C}$ and $y \in \mathbf{D}$ to $F(X, Y) \in \mathbf{E}$, and that is covariant in $X \in \mathbf{C}$ and contravariant in Y. In terms of cartesian product and duality, this can be expressed simply as a functor

$$F : \mathbf{C} \times \mathbf{D}^{op} \to \mathbf{E}$$

- **Example of mixed variance functor.** An example of a mixed variance functor is the following function in two variables on the category of sets: $F(X, Y) = X^Y$. This associates to each two sets X and Y the set X^Y of all functions from Y to X.

  Given $f : X \to X'$, there is a natural way to define

$$F(f, Y) = f^Y : X^Y \to X'^Y,$$

  namely $(f^Y)(s) = f \circ s$, where $s : Y \to X$. So F is covariant in X.

  Also, given $g : Y \to Y'$, there is a natural way to define

$$F(X, g) = X^g : X^{Y'} \to X^Y,$$

  namely $(X^g)(s) = t \circ g$, where $t : Y' \to X$. So F is contravariant in Y.

  In both variables at the same time, we have

$$F(f, g) = f^g : X^{Y'} \to X'^Y$$

  given by $f^g(t) = f \circ t \circ g$. Check that this really preserves composition and identities!

- **Some functors and non-functors on groups.**
  - To each group G, we can associate its *commutator subgroup* $[G,G]$, which is the subgroup generated by the elements $xyx^{-1}y^{-1}$, where $x,y \in G$. Suppose $f : G \to H$ is a group homomorphism. Then f, restricted to the subgroup $[G,G]$, is a group homomorphism $f : [G,G] \to [H,H]$. Namely, for each generator $xyx^{-1}y^{-1} \in [G,G]$, we have $f(xyx^{-1}y^{-1}) = f(x)f(y)f(x^{-1})f(y^{-1})$, which is an element of $[H,H]$. Checking that identities and composition are preserved is trivial in this case. Hence, the assignment $F(G) = [G,G]$ and $F[f] = f|_{[G,G]}$ defines a functor.
  - To each group G, we can associate its *center* $Z(G) = \{g \in G \mid \forall h \in G.gh = hg\}$. Then $Z(G)$ is an abelian group. Do we obtain a functor Z : **Grp** $\to$ **Ab**? As you will show in the homework, this is not the case: Z cannot be extended to morphisms to obtain a functor.

1.4 Lecture 4, Sep 22, 2005

- **Composition of functors.** If $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$ are functors, then so is $G \circ F : \mathbf{C} \to Ee$, defined by $G \circ F(A) = G(F(A))$ and $G \circ F(f) = G(F(f))$. Further, $\mathrm{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ is a functor.

- **The category of small categories.** The category **Cat** has as objects small categories (i.e., those whose objects form a set), and as morphisms all functors.

- **Isomorphism of categories.** A functor $F : \mathbf{C} \to \mathbf{D}$ is an *isomorphism of categories* if there exists $G : \mathbf{D} \to \mathbf{C}$ such that $F \circ G = \mathrm{id}_{\mathbf{C}}$ and $G \circ F = \mathrm{id}_{\mathbf{D}}$. In other words, if F is an isomorphism in **Cat**. Equivalently: $F : \mathbf{C} \to \mathbf{D}$ is an isomorphism of categories if $F : |\mathbf{C}| \to |\mathbf{D}|$ is one-to-one and onto objects, and for all $A, B \in \mathbf{C}$, $F : \mathbf{C}(A,B) \to \mathbf{D}(FA, FB)$ is one-to-one and onto.

- **Full and faithful functors.** A functor $F : \mathbf{C} \to \mathbf{D}$ is *faithful* if for all A, B, $F : \mathbf{C}(A,B) \to \mathbf{D}(FA, FB)$ is one-to-one. F is *full* if for all A, B, $F : \mathbf{C}(A,B) \to \mathbf{D}(FA, FB)$ is onto. Note that the properties of faithfulness and fullness only take into account the action of F on hom-sets, not on objects (i.e., F may or may not be one-to-one or onto objects).

- **Examples.**

  - The forgetful functor F : **Grp** $\to$ **Set** from groups to sets is faithful. Because: two group homomorphisms $f, g : G \to H$ satisfy $f = g$ (as group homomorphisms in **Grp**) if and only if they satisfy $f = g$ (as functions in **Set**). However, F is not full, because there are some functions $f : G \to H$ that are not (the image under F of) a group homomorphism.
  - The forgetful functor F : **Ab** $\to$ **Grp** from abelian groups to groups is full and faithful. Because: being abelian is a property of groups, not of homomorphism. If G, H are two abelian groups, then $f : G \to H$ is a “homomorphism of abelian groups” if and only if it is a “homomorphism of groups”.

- **Subcategories.** Let $\mathbf{C}$ be a category. A *subcategory* $\mathbf{D}$ of $\mathbf{C}$ consists of:
  - a subclass $|\mathbf{D}| \subseteq |\mathbf{C}|$ of the objects of $\mathbf{C}$, and
  - for all $A, B \in \mathbf{D}$, a subset $\mathbf{D}(A,B) \subseteq \mathbf{C}(A,B)$, such that
  - $A \in |\mathbf{D}|$ implies $\mathrm{id}_A \in \mathbf{D}(A,A)$, and
  - $A, B, C \in |\mathbf{D}|$ and $f \in \mathbf{D}(A,B)$ and $g \in \mathbf{D}(B,C)$ implies $g \circ f \in \mathbf{D}(A,C)$.

  Note that a subcategory is not just a category $\mathbf{D}$ whose class of objects and morphisms is contained in those of $\mathbf{C}$. Being a subcategory also means that the operations (identities and composition) are as in $\mathbf{C}$.

- **Inclusion functor, full subcategory.** If $\mathbf{D}$ is a subcategory of $\mathbf{C}$, then the *inclusion functor* $F_{incl} : \mathbf{D} \hookrightarrow \mathbf{C}$ if given by $F_{incl}(A) = A$ and $F_{incl}(f) = f$. Inclusion functors are always faithful. If F_{incl} is also full, then we say that $\mathbf{D}$ is a *full subcategory* of $\mathbf{C}$. Equivalently, $\mathbf{D} \subseteq \mathbf{C}$ is a full subcategory if for all $A, B \in \mathbf{D}$, $\mathbf{D}(A,B) = \mathbf{C}(A,B)$.

- **Examples.** **Ab** is a full subcategory of **Grp**. $\mathbf{Set}_{fin}$, the category of finite sets, is a full subcategory of **Set**. $\mathbf{Set}^{inj}$, the category of sets and injective functions, is a subcategory of **Set**, but it is not full.

- **Lluf functors and subcategories.** A functor $F : \mathbf{D} \to \mathbf{C}$ is lluf (“full” spelled backwards) if F is onto objects, i.e., for all $A \in |\mathbf{C}|$, there exists some $B \in \mathbf{D}$ such that $A = F(B)$. We say that a subcategory $\mathbf{D} \subseteq \mathbf{C}$ is lluf if its inclusion functor if lluf, equivalently, if $|\mathbf{D}| = |\mathbf{C}|$.

- **Examples of lluf subcategories.** $\mathbf{Set}^{inj}$, from the previous example, is a lluf subcategory of **Set**. $\mathbf{Set}^{surj}$, the category of sets and surjective functions, is also a lluf subcategory of **Set**.

- **Natural transformations.** Let $\mathbf{C}, \mathbf{D}$ be categories, and let $F, G : \mathbf{C} \to \mathbf{D}$ be functors. A *natural transformation* $\eta : F \to G$ is given by the following:

  – for each object $A \in |\mathbf{C}|$, a choice of morphism $\eta_A : FA \to GA$, such that the following diagram commutes for all $f : A \to B$:

$$\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\eta_B"] & GB. \end{tikzcd}$$

- **Examples of natural transformations.** It is helpful to think of a functor F as an "object" FA, which depends on a "set parameter" A. We can therefore think of a natural transformation as a "morphism" $\eta_A : FA \to GA$, parameterized by a set A. Naturality means that this morphism changes "consistently" as A changes.

  – Consider the singleton operation $\mathrm{sing}_X : x \mapsto \{x\}$ of type $X \to \mathscr{P}X$. This operation is defined for any set X. Moreover, it is "natural in X", in the sense that for all $f : X \to Y$, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\mathrm{sing}_X"] \arrow[d, "f"'] & \mathscr{P}X \arrow[d, "\mathscr{P}f"] \\ Y \arrow[r, "\mathrm{sing}_Y"] & \mathscr{P}Y. \end{tikzcd}$$

  Indeed, we can check that for any $x \in X$,

$$\begin{array}{rcl} (\mathscr{P}f) \circ (\mathrm{sing}_X)(x) & = & (\mathscr{P}f)(\mathrm{sing}_X(x)) \\ & = & (\mathscr{P}f)(\{x\}) \\ & = & \{fx\} \\ & = & \mathrm{sing}_Y(fx) \\ & = & (\mathrm{sing}_Y) \circ f(x). \end{array}$$

  Therefore, $\mathrm{sing}_X : X \to \mathscr{P}X$ is a natural transformation in the parameter X, also written $\mathrm{sing} : \mathrm{id}_{\mathbf{Set}} \to \mathscr{P}$.

  – Consider the "flattening" operation $\mathrm{flat}_X : \mathscr{P}(\mathscr{P}X) \to \mathscr{P}X$, which is defined by

$$\mathrm{flat}_X(\mathcal{U}) = \bigcup_{U \in \mathcal{U}} U.$$

  Here is an example: $\mathrm{flat}_X\{\{x,y\},\{z\},\{w,z\}\} = \{x,y,z,w\}$. This operation is parametric in a set X. It is also a natural transformation, in the sense that the following diagram commutes:

$$\begin{tikzcd} \mathscr{P}(\mathscr{P}X) \arrow[r, "\mathrm{flat}_X"] \arrow[d, "\mathscr{P}(\mathscr{P}f)"'] & \mathscr{P}X \arrow[d, "\mathscr{P}f"] \\ \mathscr{P}(\mathscr{P}Y) \arrow[r, "\mathrm{flat}_Y"] & \mathscr{P}Y. \end{tikzcd}$$

  We omit the proof, but give an example instead: suppose $x, y, z, w \in X$, and consider the element $\mathcal{U} = \{\{x,y\},\{z\},\{w,z\}\} \in \mathscr{P}(\mathscr{P}X)$. Then

$$\begin{array}{rcl} (\mathscr{P}f) \circ (\mathrm{flat}_X)(\mathcal{U}) & = & (\mathscr{P}f)(\{x,y,z,w\}) \\ & = & \{fx, fy, fz, fw\}, \end{array}$$

  whereas

$$\begin{array}{rcl} (\mathrm{flat}_Y) \circ (\mathscr{P}(\mathscr{P}f))(\mathcal{U}) & = & (\mathrm{flat}_Y)(\{\{fx,fy\},\{fz\},\{fw,fz\}\}) \\ & = & \{fx, fy, fz, fw\}. \end{array}$$

  So indeed, the two operations are the same in this example and the diagram commutes. Therefore, $\mathrm{flat}_X : \mathscr{P}(\mathscr{P}X) \to \mathscr{P}X$ is a natural transformation in the parameter X, and we also write

$$\mathrm{flat} : \mathscr{P} \circ \mathscr{P} \to \mathscr{P}$$

  – Consider the operation $\pi_X : X \times A \to X$, given by $\pi_X(x,a) = x$. Let us think of the set X as a parameter and of A as constant. Then the π_X operation is natural in X, namely, for all $f : X \to Y$, we have

$$\begin{tikzcd} X \times A \arrow[r, "\pi_X"] \arrow[d, "f \times A"'] & X \arrow[d, "f"] \\ Y \times A \arrow[r, "\pi_Y"] & Y. \end{tikzcd}$$

  Indeed, this is shown by a simple calculation:

$$\begin{array}{l} f(\pi_X(x,a)) = f(x) \\ \pi_Y((f \times A)(x,a)) = \pi_Y(f(x),a) = f(x). \end{array}$$

1.5 Lecture 5, Sep 27, 2005

- **Natural transformations, 2-categorical notation.** Recall that a functor $F : \mathbf{C} \to \mathbf{D}$ is between two *categories*, but a natural transformation $\eta : F \to G$ is between two *functors*. Sometimes, a good way to picture a natural transformation is as a two-dimensional "cell" connecting two functors, like this:

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathbf{D} \end{tikzcd}$$

- **Composition of natural transformations.** If $F, G, H : \mathbf{C} \to \mathbf{D}$ are three functors, and $\eta : F \to G$ and $\eta' : G \to H$ are natural transformations, then $\eta' \circ \eta : F \to H$ is a natural transformation, defined by

$$\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[rr, bend right, "(\eta' \circ \eta)_A"'] & GA \arrow[r, "\eta'_A"] & HA \end{tikzcd}$$

  Proof: to show that $\eta' \circ \eta$ is natural, let $f : A \to B$, and consider

$$\begin{tikzcd} FA \arrow[r, "\eta_A"'] \arrow[rr, bend left, "(\eta' \circ \eta)_A"] \arrow[d, "Ff"'] & GA \arrow[r, "\eta'_A"'] \arrow[d, "Gf"'] & HA \arrow[d, "Hf"'] \\ FB \arrow[r, "\eta_B"] \arrow[rr, bend right, "(\eta' \circ \eta)_B"'] & GB \arrow[r, "\eta'_B"] & HB \end{tikzcd}$$

  The two squares and the two "triangles" commute, so the outer perimeter commutes as well.

- **The identity natural transformation.** Let $F : \mathbf{C} \to \mathbf{D}$ be any functor. Then $\mathrm{id}_F : F \to F$ is the identity natural transformation, defined by $(\mathrm{id}_F)_A = \mathrm{id}_{FA} : FA \to FA$, for all $A \in |\mathbf{C}|$.

- **Natural isomorphism.** Let $F, G : \mathbf{C} \to \mathbf{D}$ be functors. A natural transformation $\eta : F \to G$ is a *natural isomorphism* if each component is an isomorphism, i.e., for all objects $A \in |\mathbf{C}|$, $\eta_A : FA \xrightarrow{\cong} GA$ is an isomorphism. Equivalently, $\eta : F \to G$ is a natural isomorphism if and only if there exists a natural transformation $\eta' : G \to F$ such that $\eta' \circ \eta = \mathrm{id}_F$ and $\eta \circ \eta' = \mathrm{id}_G$.

- **Exercise.** Prove the previous statement, i.e., a natural transformation η is invertible if and only if η_A is invertible for every A.

- **Example of natural isomorphism.** Let $2 = \{0, 1\}$ be some two-element set, and consider the following two functors $F, G : \mathbf{Set}^{op} \to \mathbf{Set}$. $FX = \mathscr{P}X$ is the contravariant power set functor, and $GX = 2^X$ is the contravariant functor mapping X to the set 2^X of all functions from X to 2.

  Intuitively, F and G are "the same functor", yet they are not quite identical. What we can say is that the sets FX and GX have the same cardinality, i.e., they are isomorphic sets. We now claim that the two functors are in fact *naturally* isomorphic.

  The isomorphism between $\mathscr{P}X$ and 2^X is well-known: it maps each subset of X to its characteristic function. More precisely: $\eta_X : \mathscr{P}X \to 2^X$ is given by

$$\eta_X(U) = \chi_U, \text{ where } \chi_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

  Clearly, η_X is an isomorphism; its inverse $\eta_X^{-1} : 2^X \to \mathscr{P}X$ is given by $\eta_X^{-1}(\chi) = \{x \in X \mid \chi(x) = 1\}$.

  What is left to check is that η_X is natural, i.e., that the following diagram commutes, for all $f : X \to Y$:

$$\begin{tikzcd} \mathscr{P}Y \arrow[r, "\eta_Y"] \arrow[d, "\mathscr{P}f"'] & 2^Y \arrow[d, "2^f"] \\ \mathscr{P}X \arrow[r, "\eta_X"] & 2^X \end{tikzcd}$$

  Recall that for $V \in \mathscr{P}Y$, $(\mathscr{P}f)(V) = f^{-1}[V] = \{x \in X \mid f(x) \in V\}$, and for $s \in 2^Y$, $2^f(s) = s \circ f$. Let $V \subseteq Y$ and $x \in X$; we calculate

$$\begin{aligned} 2^f \circ \eta_Y(V)(x) &= 2^f(\eta_Y(V))(x) \\ &= 2^f(\chi_V)(x) \\ &= \chi_V \circ f(x) \\ &= \chi_V(f(x)) \\ &= \begin{cases} 1 & \text{if } f(x) \in V \\ 0 & \text{if } f(x) \notin V \end{cases} \end{aligned}$$

Also,

$$\begin{array}{rcl}
\eta_X \circ \mathscr{P}f(V)(x) & = & \eta_X(\mathscr{P}f(V))(x) \\
& = & \eta_X(f^{-1}[V])(x) \\
& = & \chi_{f^{-1}[V]}(x) \\
& = & \begin{cases} 1 & \text{if } x \in f^{-1}[V] \\ 0 & \text{if } x \notin f^{-1}[V] \end{cases}
\end{array}$$

So $2^f \circ \eta_Y = \eta_X \circ \mathscr{P}f$, and the diagram commutes.

- **Equivalence of categories.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be functors. We recall that this situation is an *isomorphism of categories* if $G \circ F = \mathrm{id}_{\mathbf{C}}$ and $F \circ G = \mathrm{id}_{\mathbf{D}}$. A weaker, but more useful notion is that of an *equivalence of categories*: in this case, we only require that the functor $G \circ F$ is *naturally isomorphic* to $\mathrm{id}_{\mathbf{C}}$, and that $F \circ G$ is *naturally isomorphic* to $\mathrm{id}_{\mathbf{D}}$.

  More precisely, an *equivalence of categories* $\mathbf{C}$ and $\mathbf{D}$ is given by a 4-tuple (F, G, η, ϵ), where $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ are functors, and $\eta : G \circ F \xrightarrow{\cong} \mathrm{id}_{\mathbf{C}}$ and $\epsilon : F \circ G \xrightarrow{\cong} \mathrm{id}_{\mathbf{D}}$ are natural isomorphisms.

- **Remark.** We say that the categories $\mathbf{C}$ and $\mathbf{D}$ are *equivalent* if there exists an equivalence between them. Note, however, that the actual equivalence need not be unique. Not only could there be more than one pair of functors (F, G) establishing the equivalence, but for each such pair (F, G), there could in general be more than one pair of natural isomorphisms (η, ϵ).

- **Example of equivalence of categories.** For each natural number $n \geqslant 0$, let us fix some particular n-element set $\underline{n}$, for example,

  $$\underline{n} := \{0, 1, \ldots, n-1\}.$$

  Sets of this form are also called *finite ordinals*. Let **FinOrd** be the category whose objects are finite ordinals, and whose morphisms are all functions between them. Note that **FinOrd** has countably many objects and morphisms.

  On the other hand, let **Set**$_{\mathrm{fin}}$ be the category of *all* finite sets and all functions between them.

  There is a sense in which **FinOrd** and **Set**$_{\mathrm{fin}}$ are "the same": specifically, each object of **FinOrd** is "isomorphic to" some object of **Set**$_{\mathrm{fin}}$, and vice versa. On the other hand, clearly these categories are not isomorphic, since **FinOrd** has countably many objects, whereas **Set**$_{\mathrm{fin}}$ has a proper class. Instead, the categories **FinOrd** and **Set**$_{\mathrm{fin}}$ are equivalent, as we will now show.

Note that **FinOrd** is a subcategory of **Set**$_{\mathrm{fin}}$; so let $F : \mathbf{FinOrd} \hookrightarrow \mathbf{Set}_{\mathrm{fin}}$ be the inclusion functor $F(\underline{n}) = \underline{n}$, $F(f) = f$.

We define a functor $G : \mathbf{Set}_{\mathrm{fin}} \to \mathbf{FinOrd}$ as follows: for each finite set A, define $GA = \underline{n}$, where $n = |A|$, i.e., A has n elements. The question is how to extend G to morphisms. Given $f : A \to B$, where $|A| = n$ and $|B| = m$, we have to define $Gf : \underline{n} \to \underline{m}$, in such a way that G is a functor.

To do this, let us pick for each finite set A some (arbitrary, but fixed) bijection $\gamma_A : A \xrightarrow{\cong} \underline{n}$, where $n = |A|$. Then we define $Gf : \underline{n} \to \underline{m}$ to be the unique function making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\gamma_A"', "\cong"] & B \arrow[d, "\cong"', "\gamma_B"] \\
\underline{n} \arrow[r, "Gf"] & \underline{m}.
\end{tikzcd}$$

In symbols, we define $Gf = \gamma_B \circ f \circ \gamma_A^{-1}$. We claim that G is a functor. Indeed, we have, for all A, B, C and $f : A \to B$ and $g : B \to C$:

$$\begin{array}{rcl}
G(\mathrm{id}_A) & = & \gamma_A \circ \mathrm{id}_A \circ \gamma_A^{-1} \\
& = & \gamma_A \circ \gamma_A^{-1} \\
& = & \mathrm{id}_{GA}, \\
G(g \circ f) & = & \gamma_C \circ g \circ f \circ \gamma_A^{-1} \\
& = & \gamma_C \circ g \circ \gamma_B^{-1} \circ \gamma_B \circ f \circ \gamma_A^{-1} \\
& = & Gg \circ Gf,
\end{array}$$

so G is indeed a functor.

Next, we need natural isomorphisms $\eta : G \circ F \xrightarrow{\cong} \mathrm{id}$ and $\epsilon : F \circ G \xrightarrow{\cong} \mathrm{id}$.

We first describe $\eta_{\underline{n}} : G(F(\underline{n})) \to \underline{n}$. Note that $G(F(\underline{n})) = G(\underline{n}) = \underline{n}$ by definition of G, F. So it is tempting to define $\eta_{\underline{n}} = \mathrm{id}_{\underline{n}}$. However, this is not in general a natural transformation, as it would not make the following diagram commute:

$$\begin{tikzcd}
\underline{n} \arrow[r, "\eta_{\underline{n}}"] \arrow[d, "G(Ff) = \gamma_{\underline{m}} \circ f \circ \gamma_{\underline{n}}^{-1}"'] & \underline{n} \arrow[d, "f"] \\
\underline{m} \arrow[r, "\eta_{\underline{m}}"] & \underline{m}
\end{tikzcd}$$

Instead, the logical choice, making the above diagram commute, is $\eta_{\underline{n}} = \gamma_{\underline{n}}^{-1}$, and indeed, this therefore defines a natural isomorphism $\eta : G \circ F \xrightarrow{\cong} \mathrm{id}$.

For the converse direction, we similarly define $\epsilon_A : F(G(A)) \to A$ by $\epsilon_A = \gamma_A^{-1}$, which is the required natural isomorphism, because

$$\begin{tikzcd}
\underline{n} \arrow[r, "\epsilon_A"] \arrow[d, "F(Gf)=\gamma_B \circ f \circ \gamma_A^{-1}"'] & A \arrow[d, "f"] \\
\underline{m} \arrow[r, "\epsilon_B"] & B
\end{tikzcd}$$

Therefore, **FinOrd** and $\mathbf{Set}_{\mathrm{fin}}$ are equivalent categories.

1.6 Lecture 6, Sep 29, 2005

- **Functor category.** Let $\mathbf{C}, \mathbf{D}$ be two categories. The *functor category* $\mathbf{D}^{\mathbf{C}}$ is defined as follows: its objects are functors $F : \mathbf{C} \to \mathbf{D}$. A morphism in $\mathbf{D}^{\mathbf{C}}$ from F to G is a natural transformation $\eta : F \to G$.

- **Examples of functor categories for simple $\mathbf{C}$.**
  - Let $\mathbf{1}$ be the one-object, one-morphism category. Then $\mathbf{D}^{\mathbf{1}} \cong \mathbf{D}$.
  - Let $|\mathbf{2}|$ be the two-object discrete category. Then $\mathbf{D}^{|\mathbf{2}|} \cong \mathbf{D} \times \mathbf{D}$.
  - Let $\mathbf{2} = \bullet \to \bullet$ be the two-object, three-arrow category. Then $\mathbf{D}^{\mathbf{2}}$ has as its objects functors $F : \mathbf{2} \to \mathbf{D}$. By a homework problem, such functors are in one-to-one correspondence with the morphisms of $\mathbf{D}$, i.e., an object of $\mathbf{D}^{\mathbf{2}}$ is a morphism of $\mathbf{D}$. Moreover, a morphism $\eta : f \to g$ in $\mathbf{D}^{\mathbf{2}}$ is (in one-to-one correspondence with) a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\eta_0"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
A' \arrow[r, "\eta_1"] & B'.
\end{tikzcd}$$

    Composition in $\mathbf{D}^{\mathbf{2}}$ is given by pasting of diagrams:

$$\begin{tikzcd}
A \arrow[r, "\eta_0"] \arrow[d, "f"'] & B \arrow[r, "\epsilon_0"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, "\eta_1"] & B' \arrow[r, "\epsilon_1"] & C'.
\end{tikzcd}$$

- **Natural transformations in more than one variable.** Theorem: a family of operations is a natural transformation in multiple variables jointly if and only if it is natural in each variable separately (with the remaining variables fixed).

- **Side note: proofs by "diagram pasting".**

- **Monomorphisms, epimorphisms, split monics and epis.** (composition of monics is monic; $g \circ f$ monic implies f is monic, notation for monics and epics. Examples in **Set** of monics that are not split.)

- **Terminal objects, initial objects, zero objects, zero morphisms.** (examples in **Set**, **Grp**, sets and partial functions. Terminal object is unique up to iso).

2 Universal constructions: Products, Limits, Adjunctions

2.1 Lecture 7, Oct 4, 2005

- **Products, Definition 1** Let A, B be objects in a category $\mathbf{C}$. A *product cone* of A, B is a triple (D, π_1, π_2), where D is an object and $\pi_1 : D \to A$ and $\pi_2 : D \to B$ are morphisms (called the *projections*), satisfying the following **universal property**: for all objects C and all pairs of morphisms $f : C \to A$ and $g : C \to B$, there exists a unique $h : C \to D$ making the following diagram commute:

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, dotted, "\exists! h"] \arrow[dr, "g"] & \\
 & D \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
A & & B.
\end{tikzcd} \tag{1}$$

- **Notation.** We usually write $h = \langle f, g \rangle$ for the unique morphism making the above diagram commute.

- **Lemma: uniqueness up to isomorphism.** Let A, B be objects, and let (D, π_1, π_2) and (D', π'_1, π'_2) be two product cones. Then there exists an isomorphism $\varphi : D' \xrightarrow{\cong} D$ such that $\pi_1' = \pi_1 \circ \varphi$ and $\pi_2' = \pi_2 \circ \varphi$, i.e., such that

$$\begin{tikzcd}
& D' \arrow[ddl, "\pi_1'"'] \arrow[d, "\varphi", "\cong"'] \arrow[ddr, "\pi_2'"] & \\
& D \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
A & & B.
\end{tikzcd}$$

Proof. By the universal property of the product cone (D, π_1, π_2), there exists a morphism $\varphi = \langle \pi_1', \pi_2' \rangle : D' \to D$ making the above diagram commute. We must show that φ is an isomorphism. But by the universal property of the product cone (D', π_1', π_2'), there exists a morphism $\varphi' = \langle \pi_1, \pi_2 \rangle : D \to D'$ in the opposite direction. Then the following diagram commutes:

$$\begin{tikzcd}
& D' \arrow[ddl, "\pi_1'"'] \arrow[d, "\varphi", "\cong"'] \arrow[ddr, "\pi_2'"] & \\
& D \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[d, "\varphi'", "\cong"'] & \\
A & D' \arrow[l, "\pi_1'"] \arrow[r, "\pi_2'"'] & B.
\end{tikzcd}$$

Now let us consider morphisms $h : D' \to D'$ that make the following diagram commute:

$$\begin{tikzcd}
& D' \arrow[dl, "\pi_1'"'] \arrow[d, "h"] \arrow[dr, "\pi_2'"] & \\
A & D' \arrow[l, "\pi_1'"] \arrow[r, "\pi_2'"'] & B.
\end{tikzcd}$$

By the above, $h = \varphi' \circ \varphi$ is one such morphism, but clearly $h = \mathrm{id}_{D'}$ is another. The uniqueness part of the universal property for (D', π_1', π_2') implies that $\varphi' \circ \varphi = \mathrm{id}_{D'}$. By a similar argument, we have $\varphi \circ \varphi' = \mathrm{id}_D$, hence φ is an isomorphism as desired. $\square$

- **Notation.** Since a product D of A and B is "essentially" unique (i.e., unique up to isomorphism), we use the notation $D = A \times B$. However, this is slightly misleading, as a product not only involves an object, but also the two projections.

- **Products, Definition 2.** Let A, B be objects in a category $\mathbf{C}$. A *cone* over A, B is a triple (E, p_1, p_2), where E is an object and $p_1 : E \to A$ and $p_2 : E \to B$ are morphisms. (Note that a cone is not required to satisfy any further properties).

  If (E, p_1, p_2) and (E', p_1', p_2') are two cones over A, B, then a *morphism of cones* is a morphism $f : E \to E'$ such that the following diagram commutes:

$$\begin{tikzcd}
& E \arrow[ddl, "p_1"'] \arrow[d, "f"] \arrow[ddr, "p_2"] & \\
& E' \arrow[dl, "p_1'"] \arrow[dr, "p_2'"'] & \\
A & & B.
\end{tikzcd}$$

  Definition. A *product cone* is a **terminal object** in the category of cones and cone morphisms.

- **Equivalence of Definitions 1+2.** Definition 2 is clearly equivalent to Definition 1; the universal property required of a product cone has simply been restated in terms of the universal property of a terminal object. The requirement that certain diagrams commute has been neatly separated from the universal property by packaging it into the definition of a morphism of cones.

- **Advantage of Definition 2.** Since we already know that terminal objects are unique up to isomorphism, it immediately follows that product cones are unique up to isomorpism of cones. So by stating the definition in a particular way, we are able to reuse a previous lemma instead of reproving it in a new context.

- **Products, Definition 3.** Let A, B be objects in a category $\mathbf{C}$. A *product structure* of A, B is given by a 4-tuple $(D, \pi_1, \pi_2, \langle -, - \rangle)$, where D is an object, $\pi_1 : D \to A$ and $\pi_2 : D \to B$ are morphisms, and $\langle -, - \rangle$ is a family of operations

$$\langle -, - \rangle_C : \mathbf{C}(C, A) \times \mathbf{C}(C, B) \to \mathbf{C}(C, D),$$

subject to the following three **equations**, for all C, $f : C \to A$, $g : C \to B$, and $h : C \to D$:

$$\begin{array}{ll} (P1) & \pi_1 \circ \langle f, g\rangle = f \\ (P2) & \pi_2 \circ \langle f, g\rangle = g \\ (P3) & \langle \pi_1 \circ h, \pi_2 \circ h\rangle = h \end{array}$$

- **Discussion.** The universal property of Definition 1 has been replaced by an *operation* (namely, $(f, g) \mapsto \langle f, g\rangle$) and *equations*. In this sense, Definition 3 is an *algebraic* definition.

- **Equivalence of Definitions 1+3.** Given a product cone (D, π_1, π_2) satisfying the universal property of Definition 1, we can let $\langle f, g\rangle$ be the unique morphism h making diagram (1) commute. Equations $(P1)$ and $(P2)$ follow directly from diagram (1). Equation $(P3)$ follows from uniqueness: in the diagram

$$\begin{tikzcd} & C \arrow[ddl, "\pi_1\circ h"'] \arrow[d, dotted] \arrow[ddr, "\pi_2\circ h"] & \\ & D \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ A & & B, \end{tikzcd}$$

the dotted arrow can be either h or $\langle \pi_1 \circ h, \pi_2 \circ h\rangle$; since either one makes the diagram commute, by uniqueness, $(P3)$ follows.

Conversely, given a product structure $(D, \pi_1, \pi_2, \langle -, -\rangle)$ in the sense of Definition 3, we claim that (D, π_1, π_2) satisfies the universal property of Definition 1. Indeed, given any C and $f : C \to A$, $g : C \to B$, we can define $h = \langle f, g\rangle : C \to D$, and this choice will make diagram (1) commute by $(P1)$ and $(P2)$. For uniqueness, suppose $k : C \to D$ is another morphism such that

$$\begin{tikzcd} & C \arrow[ddl, "f"'] \arrow[d, "k"] \arrow[ddr, "g"] & \\ & D \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ A & & B. \end{tikzcd}$$

Then

$$k \overset{(P3)}{=} \langle \pi_1 \circ k, \pi_2 \circ k\rangle = \langle f, g\rangle = h.$$

This proves uniqueness and therefore the universal property. Finally, it is clear that the two constructions are mutually inverse, establishing the equivalence of the two definitions. □

2.2 Lecture 8, Oct 6, 2005

- **Remark.** Equations $(P1)$–$(P3)$ are equivalent to the following four equations, for all C, C' and $f : C \to A$, $g : C \to B$, $k : C' \to C$:

$$\begin{array}{ll} (P1) & \pi_1 \circ \langle f, g\rangle = f \\ (P2) & \pi_2 \circ \langle f, g\rangle = g \\ (P3a) & \langle \pi_1, \pi_2\rangle = \mathrm{id}_D \\ (P3b) & \langle f \circ k, g \circ k\rangle = \langle f, g\rangle \circ k \end{array}$$

Proof. Assume $(P1)$–$(P3)$ hold. Clearly, $(P3a)$ follows by setting $h = \mathrm{id}_D$. Also, $(P3b)$ follows like this:

$$\langle f \circ k, g \circ k\rangle \overset{(P1,P2)}{=} \langle \pi_1 \circ \langle f, g\rangle \circ k, \pi_2 \circ \langle f, g\rangle \circ k\rangle \overset{(P3)}{=} \langle f, g\rangle \circ k.$$

Conversely, $(P3a)$ and $(P3b)$ clearly imply $(P3)$ by letting $f = \pi_1$, $g = \pi_2$, and $k = h$. □

- **Products, Definition 4** Let A, B be objects in a category $\mathbf{D}$. A *product structure* of A, B is given by a pair (D, φ), where D is an object, and φ is a **natural isomorphism** of hom-sets

$$\varphi_C : \mathbf{C}(C, A) \times \mathbf{C}(C, B) \xrightarrow{\cong} \mathbf{C}(C, D).$$

□

- **Discussion.** Definition 4 is the most succinct definition of products so far. The reason it is so short is that a lot of information is contained in the words "natural" (this amounts to an equation) and "isomorphism" (this amounts to the existence of an inverse function and includes some equations).

Note that naturality means "naturality in C" (since A, B, and D are fixed). Both $\mathbf{C}(C, A) \times \mathbf{C}(C, B)$ and $\mathbf{C}(C, D)$ are contravariant functors in C

(how?). Naturality therefore means that the following diagram commutes, for all $k : C' \to C$:

$$\begin{tikzcd} \mathbf{C}(C,A) \times \mathbf{C}(C,B) \arrow[r, "\varphi_C"] \arrow[d, "\mathbf{C}(k,A)\times\mathbf{C}(k,B)"'] & \mathbf{C}(C,D) \arrow[d, "\mathbf{C}(k,D)"] \\ \mathbf{C}(C',A) \times \mathbf{C}(C',B) \arrow[r, "\varphi'_C"] & \mathbf{C}(C',D). \end{tikzcd}$$

Concretely, this means that for any $f : C \to A$, $g : C \to B$, and $k : C' \to C$,

$$\varphi_C(f \circ k, g \circ k) = \varphi_C(f, g) \circ k. \tag{2}$$

- **Equivalence of Definitions 3+4.** Given a product structure (D, φ) in the sense of Definition 4, we define a product structure $(D, \pi_1, \pi_2, \langle -, - \rangle)$ in the sense of Definition 3 as follows: setting $C = D$ yields an isomorphism of hom-sets

  $$\varphi_D : \mathbf{C}(D,A) \times \mathbf{C}(D,B) \xrightarrow{\cong} \mathbf{C}(D,D).$$

  Let (π_1, π_2) be the unique pair of morphisms such that $(\pi_1, \pi_2) = \varphi^{-1}(\mathrm{id}_D)$. Further, for any $f : C \to A$, $g : C \to B$, let $\langle f, g \rangle = \varphi_C(f, g) : C \to D$. We claim that $(D, \pi_1, \pi_2, \langle -, - \rangle)$ satisfies equations $(P1)$–$(P3)$.

  Using the naturality equation (2) from above, as well as the definitions of π_1, π_2, $\langle f, g \rangle$, we have:

  $$\begin{array}{rcl} \varphi_C(\pi_1 \circ \langle f,g \rangle, \pi_2 \circ \langle f,g \rangle) & \overset{(2)}{=} & \varphi_D(\pi_1,\pi_2) \circ \langle f,g \rangle \\ & = & \mathrm{id}_D \circ \langle f,g \rangle \\ & = & \langle f,g \rangle \\ & = & \varphi_C(f,g) \end{array}$$

  Since φ_C is a bijection, it follows that $\pi_1 \circ \langle f, g \rangle = f$ and $\pi_2 \circ \langle f, g \rangle = g$, so we have $(P1)$ and $(P2)$. For $(P3)$, we have

  $$\begin{array}{rcl} \langle \pi_1 \circ h, \pi_2 \circ h \rangle & = & \varphi_C(\pi_1 \circ h, \pi_2 \circ h) \\ & \overset{(2)}{=} & \varphi_D(\pi_1, \pi_2) \circ h \\ & = & \mathrm{id}_D \circ h \\ & = & h. \end{array}$$

  Notice how naturality was used in the proof of each equation.

  Conversely, given a product structure $(D, \pi_1, \pi_2, \langle -, - \rangle)$ in the sense of Definition 3, we define a natural isomorphism

  $$\varphi_C : \mathbf{C}(C,A) \times \mathbf{C}(C,B) \xrightarrow{\cong} \mathbf{C}(C,D)$$

  by letting $\varphi_C(f, g) = \langle f, g \rangle$. This is natural by equation $(P3b)$ (compare this with equation (2)). To show that φ_C is a bijection, define $\varphi^{-1}(h) = (\pi_1 \circ h, \pi_2 \circ h)$; the fact that φ and φ^{-1} are inverses then follows easily from $(P1)$–$(P3)$.

  Finally, it is easily checked that the two constructions are mutually inverse, establishing the equivalence of the two definitions. □

- **Example: Products in Grp.** In the category of groups, products are given by the cartesian product of groups (with the componentwise operations). Concretely, given groups G_1, G_2, their product $G_1 \times G_2$ is defined as $G_1 \times G_2 = \{(g, h) \mid g \in G_1, h \in G_2\}$, with multiplication $(g, h) \cdot (g', h') = (gg', hh')$ and unit $e_{G_1 \times G_2} = (e_{G_1}, e_{G_2})$. The projections $\pi_j : G_1 \times G_2 \to G_j$ are defined by $\pi_j(g_1, g_2) = g_j$, and are indeed group homomorphisms. Given another group K, and group homomorphisms $f_j : K \to G_j$, we can construct $h : K \to G_1 \times G_2$ by $h(k) = (f(k), g(k))$, which is again a group homomorphism. Finally, h is the unique group homomorphism (in fact, the unique function) with $\pi_j \circ h = f_j$.

- **Exercises.**
  - **Products in Cat.** Prove that the cartesian product $\mathbf{C} \times \mathbf{D}$, defined in Lecture 3, together with the obvious functors $\pi_1 : \mathbf{C} \times \mathbf{D} \to \mathbf{C}$ and $\pi_2 : \mathbf{C} \times \mathbf{D} \to \mathbf{D}$, defines a product structure on **Cat**.
  - **Products in Top.** Prove that **Top** (or your favourite concrete category) has products.
  - **Products in a poset.** Characterize what it means for a poset, regarded as a category, to have products.
  - **Products in a poset.** Characterize what it means for a monoid, regarded as a category, to have products. Can a finite monoid have products?

2.3 Lecture 9, Oct 11, 2005

- **Equalizers: motivation.** The Cartesian Plane was invented by Descartes. In modern notation, it is a cartesian product $\mathbb{R} \times \mathbb{R}$. But Descartes also discovered that functions and relations could be visualized as *subsets* of the plane. For example, the graph of the equation $x^2 + y^2 = 1$ is a circle, given by the set

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}.$$

More generally, a real number equation in two variables is of the form $f(x, y) = g(x, y)$, for some functions $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and its graph is

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid f(x, y) = g(x, y)\}.$$

Categorically, we have the following diagram:

$$\begin{tikzcd} D \arrow[r, "d"] & \mathbb{R} \times \mathbb{R} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{R}, \end{tikzcd}$$

where $d : D \to \mathbb{R} \times \mathbb{R}$ is the inclusion map. The pair (D, d) has the property that $f \circ d = g \circ d$. Moreover, it has the following *universal property*: for any pair (E, e) of an set E and a function $e : E \to \mathbb{R} \times \mathbb{R}$, if $f \circ e = g \circ e$, then there exists a unique $h : E \to D$ such that $d \circ h = e$. In diagrams:

$$\begin{tikzcd} E \arrow[d, dotted, "\exists!\, h"'] \arrow[dr, "e"] & & \\ D \arrow[r, "d"] & \mathbb{R} \times \mathbb{R} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{R}. \end{tikzcd}$$

- **Equalizers.** Let $\mathbf{C}$ be a category, A, B objects, and $f, g : A \to B$ be given. An *equalizer* of f, g is a pair (D, d), where D is an object and $d : D \to A$ is a morphism, such that:

  (a) $f \circ d = g \circ d$, and

  (b) if (E, e) is any other pair with $e : E \to A$ and $f \circ e = g \circ e$, then there exists a unique $h : E \to D$ such that $d \circ h = e$.

$$\begin{tikzcd} E \arrow[d, dotted, "\exists!\, h"'] \arrow[dr, "e"] & & \\ D \arrow[r, "d"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B. \end{tikzcd}$$

- **Equalizers in Set.** The category **Set** has equalizers, given by $D = \{a \in A \mid f(a) = g(a)\}$ and $d : D \hookrightarrow A$ the inclusion map. Proof: given any (E, e) such that $f \circ e = g \circ e$, and let $x \in E$. We define $h(x) = e(x)$. Is this really an element of D? Yes, because $f(h(x)) = f \circ e(x) = g \circ e(x) = g(h(x))$. Clearly, $d \circ h = e$, since d is an inclusion map. Finally, is h unique? Let $k : E \to D$ be another map with $d \circ k = e$. Since d is an inclusion function, it is monic, hence left cancelable, so $h = k$.

- **Some properties of equalizers.**

  – Theorem: If (D, d) is an equalizer of $f, g : A \to B$, then d is monic. Proof: let C be any object and consider $i, j : C \to D$ such that $d \circ i = d \circ j$. We must show that $i = j$.

$$\begin{tikzcd} C \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & D \arrow[r, "d"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

  Let $c = d \circ i : C \to A$. Then clearly, $f \circ c = f \circ d \circ i = g \circ d \circ i = g \circ c$, therefore, by the universal property of (D, d), there exists a unique $h : C \to D$ such that $d \circ h = c$. But $h = i$ is one such choice, and $h = j$ is another; therefore, $i = j$ and d is monic as desired. □

  – Equalizers are unique up to isomorphism. This is trivial (or rather, the proof is the usual one), because they are defined by a universal property. □

- **Pullbacks.** Given three objects A, B, C and two morphisms $f : A \to C$ and $g : B \to C$, a pullback of f and g is a universal pair of arrows $d_a : D \to A$ and $d_b : D \to B$ such that $f \circ d_a = g \circ d_b$.

  More precisely: A *cone* over f, g is a triple (E, e_a, e_b) such that E is an object, $e_a : E \to A$ and $e_b : E \to B$ are morphisms, and $f \circ e_a = g \circ e_b$.

$$\begin{tikzcd} E \arrow[r, "e_b"] \arrow[d, "e_a"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

  A cone (D, d_a, d_b) is a *pullback* of f, g if it is universal, i.e., given any other cone (E, e_a, e_b) over f, g, there exists a unique arrow $h : E \to D$ such that

$e_a = d_a \circ h$ and $e_b = d_b \circ h$.

$$
\begin{tikzcd}
E \arrow[dr, dotted, "!h"] \arrow[drr, "e_b"] \arrow[ddr, "e_a"'] & & \\
& D \arrow[r, "d_b"'] \arrow[d, "d_a"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"] & C
\end{tikzcd}
$$

- **Notation for pullbacks.** If (D, d_a, d_b) is a pullback of $f : A \to C$ and $g : B \to C$, we sometimes write (somewhat imprecisely) $D = A \times_C B$.

- **Notation for pullbacks.** In diagrams, we often use the following notation to indicate that a given square is a pullback:

$$
\begin{tikzcd}
D \arrow[r, "d_b"] \arrow[d, "d_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"] & C
\end{tikzcd}
$$

- **Pullbacks in Set.** In **Set**, the following is a pullback of $f : A \to C$ and $g : B \to C$: let

$$
\begin{array}{l}
D = \{(x, y) \mid x \in A, y \in B, f(x) = g(y)\} \\
d_a(x, y) = x \\
d_b(x, y) = y
\end{array}
$$

  It is a simple exercise to show that this indeed satisfies the universal property.

- **Coproducts, coequalizers, pushouts, initial objects.** These are the duals of products, equalizers, pullbacks, and terminal objects, respectively.

- **Coproducts in Set.** A coproduct of A, B in **Set** is the disjoint union $A + B$, together with the injections $i_1 : A \to A + B$ and $i_2 : B \to A + B$. Indeed, given any set E and a pair of functions $f : A \to E$ and $g : B \to E$, there exists a unique function $h : A + B \to E$ such that $h \circ i_1 = f$ and $h \circ i_2 = g$. Namely, the function h defined by case distinction

$$
h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B. \end{cases}
$$

- **Exercise: Coequalizers in Set.** Given $f, g : B \to A$ in **Set**, their *coequalizer* (D, d) can be found as follows. Let $\sim$ be the smallest equivalence relation on A such that for all $x \in B$, $f(x) \sim g(x)$. Let $D = A/\sim$, and let $d : A \to D$ be the quotient map of this equivalence relation. Then (D, d) is a coequalizer of f, g. Exercise: verify that this indeed defines a coequalizer.

2.4 Lecture 10, Oct 13, 2005

- **Inverse limits.** Suppose we are given a family $\{A_i\}_{i \in \mathbb{N}}$ of objects, and a morphism $a_i : A_{i+1} \to A_i$ for every i, as shown here:

$$
\begin{tikzcd}
A_0 & A_1 \arrow[l, "a_0"'] & A_2 \arrow[l, "a_1"'] & A_3 \arrow[l, "a_2"'] & \cdots \arrow[l, "a_3"']
\end{tikzcd}
$$

  A *cone* over this diagram consists of an object E and a family of morphisms $\{e_i\}_{i \in \mathbb{N}}$ such that $e_i : E \to A_i$, and for all i, $a_i \circ e_{i+1} = e_i$.

$$
\begin{tikzcd}
& & E \arrow[dll, "e_0"'] \arrow[dl, "e_1"] \arrow[d, "e_2"] \arrow[dr, "e_3"] \arrow[drr, "\cdots"] & & \\
A_0 & A_1 \arrow[l, "a_0"] & A_2 \arrow[l, "a_1"] & A_3 \arrow[l, "a_2"] & \cdots \arrow[l, "a_3"]
\end{tikzcd}
$$

  An *inverse limit* of the diagram is a terminal cone. In other words, a cone $(D, \{d_i\})$ is an inverse limit if for any other cone $(E, \{d_i\})$, there exists a unique $h : E \to D$ such that for all i, $d_i \circ h = e_i$. The diagram gets a bit messy in this case, so I will omit the d_i and e_i labels:

$$
\begin{tikzcd}
& & E \arrow[r, dotted, "!h"] \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & D \arrow[dlll] \arrow[dll] \arrow[dl] \arrow[d, "\cdots"] \arrow[dr, "\cdots"] & \\
A_0 & A_1 \arrow[l, "a_0"] & A_2 \arrow[l, "a_1"] & A_3 \arrow[l, "a_2"] & \cdots \arrow[l, "a_3"]
\end{tikzcd}
$$

- **Inverse limits in Set.** Given a family of sets $\{A_i\}_{i \in \mathbb{N}}$ and functions $a_i : A_{i+1} \to A_i$, their inverse limit in the category of sets can be calculated as follows: Consider sequences $x = \{x_i\}_{i \in \mathbb{N}}$ of elements such that $x_i \in A_i$ for all i. We say that such a sequence is *compatible* if for all i, $x_i = a_i(x_{i+1})$. Let D be the set of all compatible sequences. For any i, let $d_i : D \to A_i$ be the function defined by $d_i(x) = x_i$. We claim that $(D, \{d_i\}_i)$ defines an inverse limit.

  The details were given in class and are omitted from these notes.

- **Arbitrary limits.** We have seen that the definitions of products, equalizers, pullbacks, and inverse limits all resemble each other. They each seem to involve some notion of *cone* and a universal property. Indeed, all of these defintions (as well as the definition of a terminal object) are special cases of a general definition of the *limit of a diagram*. Before we can define this concept precisely, we must define what a diagram is.

- **Diagrams.** Let $\mathbf{D}$ be a small category (i.e., a category whose objects form a set, as opposed to a proper class). Let $\mathbf{C}$ be any category. A *diagram modeled on* $\mathbf{D}$ in $\mathbf{C}$ is a functor $F : \mathbf{D} \to \mathbf{C}$.

- **Examples.** As shown in an earlier exercise, a diagram modeled on **1** is simply an object of $\mathbf{C}$. A diagram modeled on the two-object discrete category **2** is a pair of objects of $\mathbf{C}$. Here are some other examples:

The category $\mathbf{D}$	Typical diagram modeled on $\mathbf{D}$
$\bullet_a \qquad \bullet_b$	$F_a \qquad F_b$
$\begin{tikzcd} \bullet_a \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] \arrow[r, phantom, "\vdots"] & \bullet_b \end{tikzcd}$	$\begin{tikzcd} F_a \arrow[r, shift left, "F_i"] \arrow[r, shift right, "F_j"'] \arrow[r, phantom, "\vdots"] & F_b \end{tikzcd}$
$\begin{tikzcd} & \bullet_a \arrow[d, "i"] \\ \bullet_b \arrow[r, "j"] & \bullet_c \end{tikzcd}$	$\begin{tikzcd} & F_a \arrow[d, "F_i"] \\ F_b \arrow[r, "F_j"] & F_c \end{tikzcd}$

  As these examples show, the category $\mathbf{D}$ determines the "shape" of a diagram. A diagram modeled on $\mathbf{D}$ consists of certain objects and morphisms of $\mathbf{C}$.

- **Cone over a diagram.** Given a diagram $F : \mathbf{D} \to \mathbf{C}$, a *cone over* F consists of the following data:

  (a) An object E of $\mathbf{C}$, and

  (b) a family of morphisms $\{e_a\}_{a \in |\mathbf{D}|}$, where $e_a : E \to F_a$ (one morphism for each object a of $\mathbf{D}$),

  (c) such that for all morphisms $i : a \to b$ of $\mathbf{D}$, the following triangle commutes:

$$\begin{tikzcd} E \arrow[d, "e_a"'] \arrow[dr, "e_b"] & \\ F_a \arrow[r, "F_i"] & F_b. \end{tikzcd}$$

When there are many objects and morphisms in $\mathbf{D}$, a cone is messy to draw, so we usually restrict ourselves to drawing one or two objects and morphisms at any given time.

- **Limit cone.** Let $F : \mathbf{D} \to \mathbf{C}$ be a diagram in $\mathbf{C}$, modeled on $\mathbf{D}$. A cone $(D, \{d_a\}_a)$ over F is called a *limit cone* if it satisfies the following universal property: given any other cone $(E, \{e_a\}_a)$, there exists a unique morphism $h : D \to E$ such that for all $a \in |\mathbf{D}|$, the following triangle commutes:

$$\begin{tikzcd} E \arrow[r, dotted, "!h"'] \arrow[dr, "e_a"'] & D \arrow[d, "d_a"] \\ & F_a. \end{tikzcd} \tag{3}$$

- **Special cases.** Various limits that we have previously defined are special cases of this general definitions. The only thing that changes is the category $\mathbf{D}$. For example, a product is a limit of a diagram modeled on the two-object discrete category **2**. An equalizer is the limit of a diagram modeled on

$$\begin{tikzcd} \bullet_a \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] \arrow[r, phantom, "\vdots"] & \bullet_b. \end{tikzcd}$$

  A pullback is the limit of a diagram modeled on

$$\begin{tikzcd} & \bullet_a \arrow[d, "i"] \\ \bullet_b \arrow[r, "j"] & \bullet_c. \end{tikzcd}$$

  An inverse limit is the limit of a diagram modeled on

$$\begin{tikzcd} \bullet_0 & \bullet_1 \arrow[l, "i_0"'] & \bullet_2 \arrow[l, "i_1"'] & \bullet_3 \arrow[l, "i_2"'] & \cdots \arrow[l, "a_3"'] \end{tikzcd}$$

  And a terminal object is the limit of a diagram modeled on the empty category. (Since the empty category has no objects, there is only one such diagram, namely, the empty diagram. Its limit is a terminal object).

- **Constant diagrams and the diagonal functor.** Given a small category $\mathbf{D}$, and a fixed object A in the category $\mathbf{C}$, there is always a special diagram in $\mathbf{C}$ modeled on $\mathbf{D}$, called the *constant A diagram*: the only object occuring in this diagram is A, and the only morphisms are id_A. For example, the constant A diagram modeled on

$$\begin{tikzcd} & \bullet_a \arrow[d, "i"] \\ \bullet_b \arrow[r, "j"] & \bullet_c \end{tikzcd}$$

is

$$\begin{tikzcd} & A \arrow[d, "\mathrm{id}_A"] \\ A \arrow[r, "\mathrm{id}_A"] & A. \end{tikzcd}$$

More formally, the constant A diagram is the functor $F : \mathbf{D} \to \mathbf{C}$ defined by $F(a) = A$ for all $a \in |\mathbf{D}|$, and $F(i) = \mathrm{id}_A$ for all $i : a \to b$ in $\mathbf{D}$.

The functor $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{D}}$ that maps each A to its constant diagram is called the *diagonal functor*.

- **Definition of limits in terms of the diagonal functor.** Recall that a cone over F consists of an object E, together with a family of morphism $\{e_a\}_{a\in|\mathbf{D}|}$ such that $e_a : E \to F_a$, and such that the following triangles commute, for all $i : a \to b$ in $\mathbf{D}$:

$$\begin{tikzcd} E \arrow[d, "e_a"'] \arrow[dr, "e_b"] & \\ F_a \arrow[r, "F_i"] & F_b. \end{tikzcd}$$

An equivalent definition is obtained by replacing the triangle by a square:

$$\begin{tikzcd} E \arrow[r, "\mathrm{id}_E"] \arrow[d, "e_a"'] & E \arrow[d, "e_b"] \\ F_a \arrow[r, "F_i"] & F_b. \end{tikzcd}$$

Notice that this square precisely means that $e : \Delta E \to F$ is a natural transformation. We therefore obtain the following compact definition of a cone:

Definition. A *cone over the diagram* $F : \mathbf{D} \to \mathbf{C}$ consists of a pair (E, e), where E is an object of $\mathbf{C}$, and $e : \Delta E \to F$ is a natural transformation.

In the same style, we also obtain a definition of limit:

Definition. A *limit over the diagram* $F : \mathbf{D} \to \mathbf{C}$ consists of a pair (D, d), where D is an object of $\mathbf{C}$, and $d : \Delta D \to F$ is a natural transformation, satisfying the following universal property: for any object E and any natural transformation $e : \Delta E \to F$, there exists a unique $h : E \to D$ such that

$$\begin{tikzcd} \Delta E \arrow[d, "\Delta h"'] \arrow[dr, "e"] & \\ \Delta D \arrow[r, "d"] & F \end{tikzcd}$$

Notice that the last diagram is a diagram of *natural transformations*. Spelled out in terms of its components, this diagram is saying that for all $a \in |\mathbf{D}|$,

$$\begin{tikzcd} E \arrow[d, "h"'] \arrow[dr, "e_a"] & \\ D \arrow[r, "d_a"] & F_a. \end{tikzcd}$$

This was precisely the universal property given earlier in (3).

2.5 Lecture 11, Oct 18, 2005

- **Product of an indexed family of sets.**
- **Products and equalizers imply all limits.**