

Propositional Logic

Propositions are made from propositional symbols, such as p, q, r , and propositional connectives such as "and", "or", "not", and "implies".

Write A, B, C for arbitrary propositions, e.g.: $A = (p \wedge q) \Rightarrow \neg p$.

A list of propositions is called a context: $\Gamma = A_1, \dots, A_n$

The fundamental object of logic is the judgement: $A_1, \dots, A_n \vdash B$.
 B is called the conclusion, and $A_1, \dots, A_n$ are called the hypotheses.

The judgement is valid if the hypotheses imply the conclusion.

Note: the purpose of logic is not to determine which propositions are true. Its purpose is to determine which propositions follow from other propositions.

Propositional Logic

"Simple" Propositions p, q, r

Composite props: $(p \wedge q)$, $(p \vee q)$, $\neg p$, $p \Rightarrow q$, $(p \wedge q) \vee \neg p$
 and or not

Write A, B, C for arbitrary (possibly compound) propositions.

The fundamental object of logic is the judgement

$$A_1, \dots, A_n \vdash B$$

↑
hypotheses

↑
turnstile

↑
conclusion

$$\Gamma \vdash B$$

Valid judgements can be derived by a small number of inference rules. (There are approximately 13 of them, depending how you count). Here are some examples.

The inference rules

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad \text{"AND" INTRODUCTION RULE}$$
$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \quad \text{"AND" ELIMINATION RULE}$$

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \quad \begin{array}{l} \text{MODUS PONENS} \\ \text{"IMPLICATION" ELIMINATION} \end{array}$$
$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \quad \text{"IMPLICATION" INTRODUCTION}$$

$$\frac{}{A_1, \dots, A_n \vdash A_i} \quad \text{"AXIOM"}$$

Here is an example of a derivation (or proof). We prove a judgement by repeatedly applying inference rules. This particular derivation proves that the judgement $\vdash p \wedge q \Rightarrow q \wedge p$ is valid. In other words, that the proposition $p \wedge q \Rightarrow q \wedge p$ follows from an empty set of hypotheses.

$$\begin{array}{c}
 \frac{}{p \wedge q \vdash p \wedge q} \text{ (ass.)} \quad \frac{}{p \wedge q \vdash p \wedge q} \text{ (ass.)} \\
 \frac{p \wedge q \vdash p \wedge q}{p \wedge q \vdash q} \text{ (}\wedge E_2\text{)} \quad \frac{p \wedge q \vdash p \wedge q}{p \wedge q \vdash p} \text{ (}\wedge E_1\text{)} \\
 \frac{p \wedge q \vdash q \quad p \wedge q \vdash p}{p \wedge q \vdash q \wedge p} \text{ (}\wedge I\text{)} \\
 \frac{p \wedge q \vdash q \wedge p}{\vdash p \wedge q \rightarrow q \wedge p} \text{ (}\rightarrow I\text{)}
 \end{array}$$