

Lecture notes, Sept 22, 2026

Recall the Brouwer-Heyting-Kolmogorov interpretation of logic. We saw the cases for implication and "and" last time. Today, we also added "or".

BROUWER, HEYTING, KOLMOGOROV

- A proof of $A \rightarrow B$ is a function that maps proofs of A to proofs of B .
- A proof of $A \wedge B$ is a pair of a proof of A and a proof of B .
- A proof of $A \vee B$ is either a proof of A or a proof of B , together with an indicator of which of the two it is.

We write $X + Y$ for the disjoint union of two sets X and Y . The disjoint union means: rename all the elements of X and of Y so that they become distinct. Then take the union. For concreteness, we "rename" the elements of X as "left x ", and the elements of Y as "right y ". So:

$$X + Y = \{\text{left } x \mid x \in X\} \cup \{\text{right } y \mid y \in Y\}.$$

With these conventions, we can summarize the BHK interpretation in symbols:

$$\text{Proof } (A \rightarrow B) \cong \text{Proof } A \rightarrow \text{Proof } B$$

$$\text{Proof } (A \wedge B) \cong \text{Proof } A \times \text{Proof } B$$

$$\text{Proof } (A \vee B) \cong \text{Proof } A + \text{Proof } B$$

The Curry-Howard correspondence (or Curry-Howard isomorphism) goes one step further, and removes the word "Proof". Instead of saying that "Proof A " is the set of proofs of A , we say that the proposition A itself is the set of its proofs. So a proposition A is a type, and its elements are the proofs of A . Since the elements of Prop are types, Prop is a universe.

OLD NOTATION
 $x : \text{Proof } A$

LEAN NOTATION
 $x : A : \text{Prop}$

Prop is a universe

Under the Curry-Howard isomorphism, the BHK correspondence then becomes particularly simple:

Curry-Howard isomorphism

$$\begin{aligned} A \rightarrow B &= A \multimap B \\ A \wedge B &= A \times B \\ A \vee B &= A + B \end{aligned}$$

The slogans are: propositions are types, proofs are terms.

We also say that "proofs are programs".

This only works for "constructive" proofs. The BHK interpretation does not admit the excluded middle principle $A \vee \neg A$.