MAT 5361, TOPICS IN QUANTUM COMPUTATION, WINTER 2004
Homework 2

Problem 1. Recall that a *traced monoidal category* $\langle \mathbf{C}, \otimes, \mathrm{Tr}\rangle$ is a symmetric monoidal category $\langle \mathbf{C}, \otimes\rangle$, with symmetries $c_{A,B} : A \otimes B \to B \otimes A$, together with a family of operations

$$\mathrm{Tr}_X : \mathbf{C}(A \otimes X, B \otimes X) \to \mathbf{C}(A, B),$$

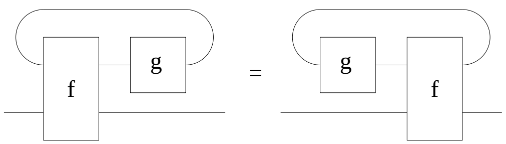
satisfying the following four axioms:

1. Naturality. $\mathrm{Tr}_X(g \otimes \mathrm{id}_X; f; h \otimes \mathrm{id}_X) = g; \mathrm{Tr}_X\, f; h$.
2. Strength. $\mathrm{Tr}_X(g \otimes f) = g \otimes \mathrm{Tr}_X\, f$.
3. Symmetry sliding. $\mathrm{Tr}_Y(\mathrm{Tr}_X(f; \mathrm{id}_B \otimes c_{XY})) = \mathrm{Tr}_X(\mathrm{Tr}_Y(\mathrm{id}_A \otimes c_{XY}; f))$.
4. Yanking. $\mathrm{Tr}_X(c_{XX}) = \mathrm{id}_X$.

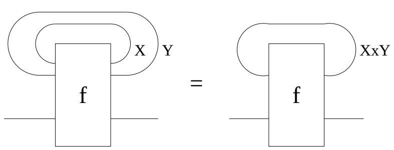
Recall from class that there is a graphical language for traced monoidal categories. The above four axioms are represented in the graphical language in Table **??**. Recall that we claimed in class that an equation follows from the above axioms if and only if the corresponding graphs in the graphical language are isomorphic.

Prove the following equations *without using this fact*, i.e., directly from the axioms. Hint: you may still reason graphically, but *only* using the transformation allowed by the above axioms and those of the symmetric monoidal structure. Translate the resulting proof into algebraic language.

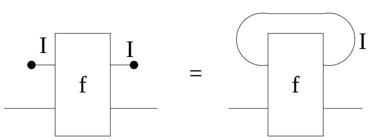
(a) General sliding. $\mathrm{Tr}_Y(f; \mathrm{id}_B \otimes g) = \mathrm{Tr}_X(\mathrm{id}_A \otimes g; f)$, where $f : A \otimes Y \to B \otimes X$, $g : X \to Y$.



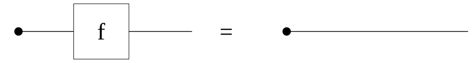
(b) Vanishing #1. $\mathrm{Tr}_X(\mathrm{Tr}_Y(f)) = \mathrm{Tr}_{X \otimes Y}(f)$, where $f : A \otimes X \otimes Y \to B \otimes X \otimes Y$.

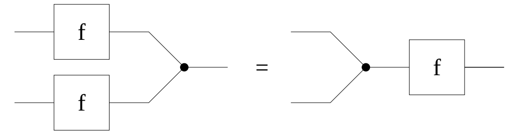


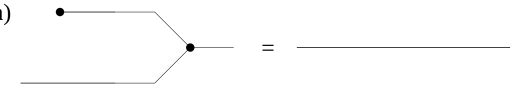
(c) Vanishing #2. $\mathrm{Tr}_I(f) = l_A; f; l_B^{-1}$, where $f : A \otimes I \to B \otimes I$, and $l_A : A \to A \otimes I$ and $l_B : B \to B \otimes I$ are the canonical isomorphisms given by the symmetric monoidal structure.

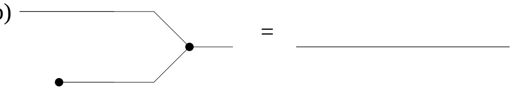


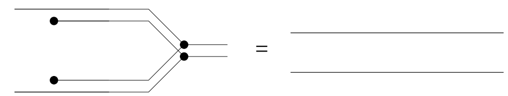
Problem 2. Recall the graphical axiomatization of categories with coproducts given in class:

1)


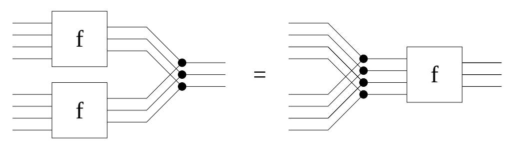
2)


3a)


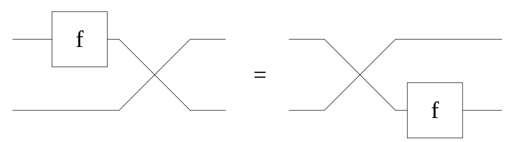
3b)


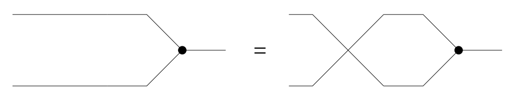
4)


Here, it is understood that each single line shown in these axioms can actually represent several parallel lines. For instance, the following is an instance of axiom 2):



Problem: derive the following 2 equations from the above axioms:





Problem 3. In class, we have defined categorical structures in terms of *constructors* and *equations*. For example, we defined products as follows:

$$\frac{}{A\times B}\qquad\frac{f:A\to B\quad g:A\to C}{\langle f,g\rangle:A\to B\times C}$$

$$\frac{}{\pi_1^{B,C}:B\times C\to B}\qquad\frac{}{\pi_2^{B,C}:B\times C\to C},$$

subject to these equations, where $f:A\to B$, $g:A\to C$, $h:A\to B\times C$:

$$\begin{aligned}&\pi_1\circ\langle f,g\rangle=f\\&\pi_2\circ\langle f,g\rangle=g\\&\langle\pi_1\circ h,\pi_2\circ h\rangle=h\end{aligned}$$

Many category theorists will instead prefer to define such structures in terms of *universal properties*. For instance, the definition of products in terms of a universal property is the following: an object P is called a *product* of B and C if there are morphisms $\pi_1:P\to B$ and $\pi_2:P\to C$ such that, for every object A and every pair of morphisms $f:A\to B$ and $g:A\to C$, there *exists a unique* morphism $h:A\to B\times C$ such that the following diagram commutes:

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\ B & P \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \end{tikzcd}$$

(a) Prove that the two definitions are, in a suitable sense, equivalent.

(b) Using the "universal" definition, prove that products are uniquely determined, i.e., if P and Q are products of B and C then $P\cong Q$. Here, we say that two objects P,Q are *isomorphic*, in symbols $P\cong Q$, if there exist morphisms $a:P\to Q$ and $b:Q\to P$ such that $a\circ b=\mathrm{id}_Q$ and $b\circ a=\mathrm{id}_P$.

Problem 4. Recall that a symmetric monoidal category $\langle\mathbf{C},\otimes\rangle$ is *closed* if there exists an operation on objects $B\multimap C$ and an isomorphism of hom-sets $\mathbf{C}(A\otimes B,C)\cong\mathbf{C}(A,B\multimap C)$, naturally in A. Prove that this definition is equivalent to the following definition in terms of a universal property:

Definition. A *closed structure* on a symmetric monoidal category $\langle\mathbf{C},\otimes\rangle$ is given by an operation on objects $B\multimap C$, and a family of morphisms $\epsilon_{B,C}:(B\multimap C)\otimes B\to C$ called the *application maps*, such that for any $f:A\otimes B\to C$, there exists a unique map $f^\dagger:A\to B\multimap C$ such that the following diagram commutes:

$$\begin{tikzcd} (B\multimap C)\otimes B \arrow[r, "\epsilon_{B,C}"] & C \\ A\otimes B \arrow[u, dotted, "f^\dagger\otimes\mathrm{id}_B"] \arrow[ur, "f"'] & \end{tikzcd}$$

Using this latter definition, also prove that, in a given symmetric monoidal category, any closed structure is uniquely determined up to isomorphism.

Problem 5. Recall the definition of the category $\mathbf{Q}'$. The objects are pairs $V=\langle\sigma,\|-\|_V\rangle$, where $\sigma=n_1,\ldots,n_s$ is a *signature*, and $\|-\|_V$ is a norm on P_σ, the set of hermitian positive matrix tuples of type σ:

$$P_\sigma=\{(A_1,\ldots,A_s)\mid A_i\in\mathbb{C}^{n_i\times n_i}\text{ and }A_i\text{ positive}\}$$

Here, a *norm* is a function $\|-\|:P_\sigma\to\mathbb{R}_+$ satisfying the following properties:

- Strictness. $\|A\|=0\Rightarrow A=0$.
- Linearity. $\|\lambda A\|=\lambda\|A\|$, for $\lambda\in\mathbb{R}_+$.
- Triangle inequality. $\|A+B\|\leqslant\|A\|+\|B\|$.
- Monotonicity. $A\sqsubseteq B\Rightarrow\|A\|\leqslant\|B\|$.
- Continuity. $A=\bigvee^{\uparrow}_i A_i\Rightarrow\|A\|=\bigvee^{\uparrow}_i\|A_i\|$, where (A_i) is an increasing sequence.

A *morphism* is just a linear, norm-non-increasing function. More precisely, a morphism from $V=\langle\sigma,\|-\|_V\rangle$ to $W=\langle\tau,\|-\|_W\rangle$ is a linear function $f:P_\sigma\to P_\tau$ such that for all $A\in P_\sigma$, $\|f(A)\|_W\leqslant\|A\|_V$.

Prove that this category has finite products and coproducts. What are they? Is there a way of making $\langle\mathbf{Q}',\oplus\rangle$ into a traced monoidal category?

Problem 6. Let $V=\langle\sigma,\|-\|_V\rangle$ and $W=\langle\tau,\|-\|_W\rangle$ be objects in the category $\mathbf{Q}'$ (see Problem 5). Recall that their tensor product is defined as $V\otimes W:=\langle\sigma\otimes\tau,\|-\|_{V\otimes W}\rangle$, where

$$\|C\|_{V\otimes W}=\inf\{\sum_i\|A_i\|_V\|B_i\|_W\mid C\sqsubseteq\sum_i A_i\otimes B_i,\text{ where }A_i\in V,\,B_i\in W\}.$$

Prove the following theorems, which were stated in class. You may use other theorems proved in class, such as the characterization of $\|-\|_{V\otimes W}$ in terms of its unit region.

(a) For $A\in V$ and $B\in W$, $\|A\otimes B\|_{V\otimes W}=\|A\|_V\otimes\|B\|_W$.

(b) $\|C\|_{V\otimes W}=0\Rightarrow C=0$. (This is part of the proof that $\|-\|_{V\otimes W}$ is indeed a norm; the only part we did not give in class).

(c) Associativity: $\|C\|_{(V\otimes W)\otimes U}=\|C\|_{V\otimes(W\otimes U)}$.

Problem 7. Let $V=\langle 2,\|-\|_{\mathrm{tr}}\rangle$, the space of hermitian positive 2×2-matrices with the *trace norm* $\|A\|_{\mathrm{tr}}=\mathrm{tr}\,A$. Recall the definition of $\|-\|_{V\otimes V}$, see Problem 6. Calculate the following quantities exactly, if possible, or else find upper and lower bounds:

$$\left\|\begin{pmatrix}2&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&2\end{pmatrix}\right\|_{\mathrm{tr}\otimes\mathrm{tr}},\qquad\left\|\begin{pmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&1\end{pmatrix}\right\|_{\mathrm{tr}\otimes\mathrm{tr}},\qquad\left\|\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1\end{pmatrix}\right\|_{\mathrm{tr}\otimes\mathrm{tr}}.$$