MATH 285, HONORS MULTIVARIABLE CALCULUS, FALL 1999

Answers to the Final Exam

Problem 1 Find an equation of the tangent plane to the surface $z + 1 = xe^y \cos z$ at the point $(1, 0, 0)$.

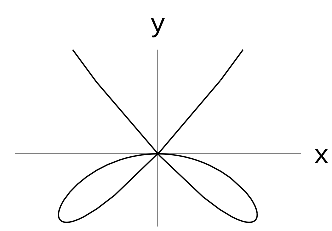
Answer: We can write the surface as $f(x, y, z) = 1$, where $f(x, y, z) = xe^y \cos z - z$. The tangent plane that we are looking for will be parallel to the level surface of f at the given point. We also know that the gradient is orthogonal to the level surface, and thus the gradient of f will be a normal vector for the desired tangent plane. We calculate $\nabla f = \langle e^y \cos z, xe^y \cos z, -xe^y \sin z - 1\rangle = \langle 1, 1, -1\rangle$. We need an equation for the plane through $\mathbf{r}_0 = (1, 0, 0)$ with normal vector $\mathbf{n} = \langle 1, 1, -1\rangle$. Such an equation is $\mathbf{r} \cdot \mathbf{n} = \mathbf{r}_0 \cdot \mathbf{n}$, or $x + y - z = 1$.

Problem 2 Calculate $\iint_D x^2 + y^2 \, dA$, where D is the unit disk $x^2 + y^2 \leqslant 1$.

Answer: This is best calculated in polar coordinates. $\iint_D x^2+y^2 \, dA = \iint_D r^2 \cdot r \, dr d\theta = \int_0^{2\pi} \int_0^1 r^3 \, dr d\theta = 2\pi \cdot \frac{1}{4} = \pi/2$.

Problem 3 The parametric equations

$$\begin{aligned} x &= t^3 - t \\ y &= t^4 - t^2 \end{aligned}$$



describe a curve which has two loops, as shown in the illustration. Calculate the area enclosed by one of the loops.

Answer: The curve passes through the origin at $t = -1$, $t = 0$, and $t = 1$. Thus, one of the loops is traced out for $0 \leqslant t \leqslant 1$ (the other loop is symmetrical). We calculate the area:

$$\begin{aligned} \int x \, dy &= \int_0^1 xy' \, dt \\ &= \int_0^1 (t^3 - t)(4t^3 - 2t) dt \\ &= \int_0^1 4t^6 - 6t^4 + 2t^2 \, dt \\ &= \left[\tfrac{4}{7}t^7 - \tfrac{6}{5}t^5 + \tfrac{2}{3}t^3\right]_0^1 \\ &= \tfrac{4}{7} - \tfrac{6}{5} + \tfrac{2}{3} \\ &= \tfrac{60-126+70}{105} \\ &= \tfrac{4}{105}. \end{aligned}$$

Problem 4 Minimize $x^2 + 4y^2 + 9z^2$ subject to the constraint $xyz = \frac{4}{3}$, for $x, y, z > 0$.

Answer: We use he method of Lagrange multipliers. Let $f(x, y, z) = x^2 + 4y^2 + 9z^2$ and $g(x, y, z) = xyz$. Since f and g are differentiable, we know that all local minima occur at points (x, y, z) that satisfy

$$\nabla f(x, y, z) = \lambda \cdot \nabla g(x, y, z),$$

for some λ. This means,

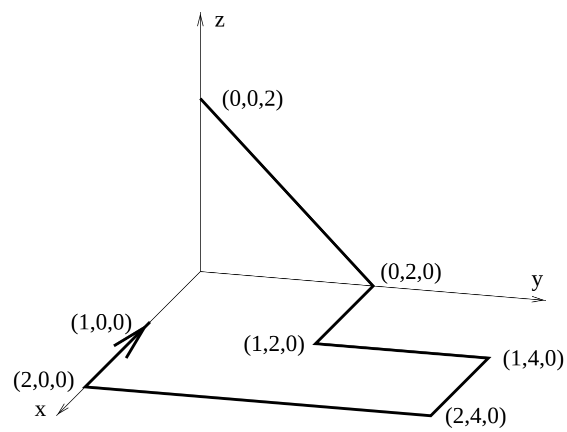
$$\langle 2x, 8y, 18z\rangle = \lambda \cdot \langle yz, xz, xy\rangle.$$

So we get $\lambda = 2x/yz = 8y/xz = 18z/xy$, and, by multiplying everything by $xyz/2$, $x^2 = 4y^2 = 9z^2$. Thus we have $y = x/2$ and $z = x/3$. We also know that the point we are looking for satisfies the constraint $xyz = 4/3$, thus $x^3/6 = 4/3$, thus $x^3 = 8$ or $x = 2$. It follows that $\langle x, y, z\rangle = \langle 2, 1, 2/3\rangle$ is the only critical point. This is indeed the global minimum.

Problem 5 Compute the outward flux of $\mathbf{F}(x, y, z) = (x^2 + \sin y)\mathbf{i} + y^2\mathbf{j} + (x - 2z)y\mathbf{k}$ through the surface of the cube $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 2$.

Answer: By the Divergence Theorem, the flux is equal to $\iiint_E \operatorname{div}\mathbf{F}\,dV$, where E is the cube in question. We calculate $\operatorname{div}\mathbf{F} = 2x + 2y - 2y = 2x$. Thus, $\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^2\int_0^2\int_0^2 2x\,dxdydz = 16$.

Problem 6 Let $\mathbf{F}(x, y, z) = (x + e^y)\mathbf{i} + (xe^y + 2ye^z)\mathbf{j} + (y^2e^z + 4z)\mathbf{k}$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve with initial point $(0, 0, 2)$ and terminal point $(1, 0, 0)$ shown in the figure.



Answer: We calculate $\operatorname{curl}\mathbf{F} = 0$ and conclude that $\mathbf{F}$ is conservative. A potential field is easily found: $\mathbf{F} = \nabla f$ where $f = x^2/2 + xe^y + y^2e^z + 2z^2$. So by the fundamental theorem for line integrals, we have $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1, 0, 0) - f(0, 0, 2) = 1.5 - 8 = -6.5$.

Problem 7 Evaluate the integral $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$, where S is the part of the sphere $x^2+y^2+z^2 = 1$ with $z \geqslant 0$, and $\mathbf{F}(x, y, z) = (z - y)\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} + e^{xyz}\mathbf{k}$. Choose the upward orientation for S.

Answer: By Stokes' Theorem, $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot\mathbf{r}$, where C is the boundary of S. Thus, C is the curve $x^2 + y^2 = 0$ with $z = 0$, oriented in the positive direction. We parameterize C with $0 \leqslant t \leqslant 2\pi$ as

$$\begin{aligned} x(t) &= \cos t \\ y(t) &= \sin t \\ z(t) &= 0. \end{aligned}$$

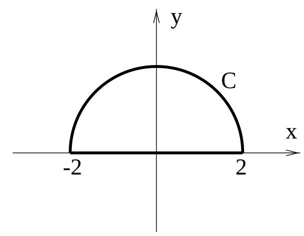
We calculate $\mathbf{r}'(t) = \langle -\sin t, \cos t, 0\rangle$ and $\mathbf{F}(x(t), y(t), z(t)) = (0-\sin t)\mathbf{i} + \frac{\cos t}{\cos^2 t+\sin^2 t}\mathbf{j} + e^{\cos t\cdot\sin t\cdot 0}\mathbf{k} = \langle -\sin t, \cos t, 1\rangle$. Now we can calculate the line integral: $\int_C \mathbf{F}\cdot\mathbf{r} = \int_0^{2\pi}\mathbf{F}\cdot\mathbf{r}'dt = \int_0^{2\pi}\langle -\sin t, \cos t, 1\rangle\cdot\langle -\sin t, \cos t, 0\rangle dt = \int_0^{2\pi} 1dt = 2\pi$.

Problem 8 Assume that f and g are scalar fields with continuous second-order partial derivatives. Prove that $\operatorname{div}(\nabla f \times \nabla g) = 0$.

Answer:

$$\begin{aligned}
\operatorname{div}(\nabla f \times \nabla g) &= \operatorname{div}(\langle f_x, f_y, f_z\rangle \times \langle g_x, g_y, g_z\rangle) \\
&= \operatorname{div}(\langle f_y g_z - f_z g_y, f_z g_x - f_x g_z, f_x g_y - f_y g_x\rangle) \\
&= \tfrac{\partial}{\partial x}(f_y g_z - f_z g_y) + \tfrac{\partial}{\partial y}(f_z g_x - f_x g_z) + \tfrac{\partial}{\partial z}(f_x g_y - f_y g_x) \\
&= f_{yx} g_z + f_y g_{zx} - f_{zx} g_y - f_z g_{yx} \\
&\quad + f_{zy} g_x + f_z g_{xy} - f_{xy} g_z - f_x g_{zy} \\
&\quad + f_{xz} g_y + f_x g_{yz} - f_{yz} g_x - f_y g_{xz} \\
&= 0
\end{aligned}$$

Problem 9 Use Green's Theorem to evaluate the line integral $\int_C 4xy - 4y\,dx + 2x^2\,dy$, where C is the positively oriented curve that consists of the line segment from $(-2, 0)$ to $(2, 0)$ and the top half of the circle $x^2 + y^2 = 4$.



Answer: Let $\langle P, Q\rangle = \langle 4xy - 4y, 2x^2\rangle$. By Green's theorem, the line integral $\int_C 4xy - 4y\,dx + 2x^2\,dy$ is equal to

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 4x - (4x - 4)\,dA = \iint_D 4\,dA,$$

where D is the region enclosed by C. The answer is 4 times the area of D, or 8π.