MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999
Answers to Problem Set 1

Problem 1.4 Assume that $x, y \in B$. Then $\{x\} \subseteq B$ and $\{x, y\} \subseteq B$, and thus $\{x\} \in \mathcal{P}B$ and $\{x, y\} \in \mathcal{P}B$. It follows that $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$, and thus $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$.

Problem 1.5 We have:

1. $V_0 = \emptyset$,
2. $V_1 = \mathcal{P}V_0 = \{\emptyset\}$,
3. $V_2 = \mathcal{P}V_1 = \{\emptyset, \{\emptyset\}\}$,
4. $V_3 = \mathcal{P}V_2 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$, etc.

If $A = \{\{\emptyset\}\}$ and $B = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, then we see that $A \subseteq V_2$ but $A \not\subseteq V_1$. Thus the rank of A is 2. Similarly, $B \subseteq V_3$ but $B \not\subseteq V_4$, thus the rank of B is 3.

Problem 2.2 Let $A = \{\{a\}, \{b\}\}$ and $B = \{\{a, b\}\}$, for some $a \neq b$.

Problem 2.3 Let $a \in A$. Then for any $x \in a$, by definition of union, we have $x \in \bigcup A$. Thus, $a \subseteq \bigcup A$.

Problem 2.4 Suppose $A \subseteq B$. We want to show $\bigcup A \subseteq \bigcup B$. So take any $x \in \bigcup A$. Then, by definition of $\bigcup A$, there is some $a \in A$ with $x \in a$. But because $A \subseteq B$, we also have $a \in B$, and thus $x \in \bigcup B$ by definition of $\bigcup B$. Since x was arbitrary, this shows $\bigcup A \subseteq \bigcup B$.

Problem 2.5 Assume every member of $\mathcal{A}$ is a subset of B. We want to show that $\bigcup \mathcal{A} \subseteq B$. So take any $x \in \bigcup \mathcal{A}$. If suffices to show that $x \in B$. By definition of $\bigcup \mathcal{A}$, there is some $a \in \mathcal{A}$ with $x \in a$. By assumption, $a \subseteq B$. It follows that $x \in B$ as desired.

Problem 2.6

(a) By extensionality, it is enough to show that each element of either of these sets is also in the other.

Suppose $x \in \bigcup \mathcal{P}A$. We must show that $x \in A$. By definition of union, there is an $a \in \mathcal{P}A$ with $x \in a$. But, by definition of the power set, $a \subseteq A$, and hence $x \in A$.

Conversely, take any $x \in A$. Then we have $x \in \{x\}$ and also $\{x\} \in \mathcal{P}A$. Thus, by definition of union, $x \in \bigcup \mathcal{P}A$.

(b) To show that $A \subseteq \mathcal{P} \bigcup A$, take any $x \in A$. By Problem 2.3, we have $x \subseteq \bigcup A$, and thus $x \in \mathcal{P} \bigcup A$, as desired.

Equality does not in general hold: For instance, let A by any set with $\emptyset \notin A$. Since $\emptyset \in \mathcal{P} \bigcup A$, we have $\mathcal{P} \bigcup A \not\subseteq A$. However, $A = \mathcal{P} \bigcup A$ holds if A is a powerset: if $A = \mathcal{P}B$, for some B, then $\mathcal{P} \bigcup A = \mathcal{P} \bigcup \mathcal{P}B = \mathcal{P}B = A$, where the second equality holds by part (a), applied to B.

Problem 2.7

(a) Each of the following statements is equivalent to the next, by definition of $\cap$, $\subseteq$, and $\mathcal{P}$: $a \in \mathcal{P}A \cap \mathcal{P}B$. $a \in \mathcal{P}A$ and $a \in \mathcal{P}B$. $a \subseteq A$ and $a \subseteq B$. Any $x \in a$ is in A and B. Any $x \in a$ is in $A \cap B$. $a \subseteq A \cap B$. $a \in \mathcal{P}(A \cap B)$. So the desired equality follows by extensionality.

(b) Note that $A \subseteq A \cup B$, by definition of $\subseteq$ and $\cup$. By Problem 1.3, this implies $\mathcal{P}A \subseteq \mathcal{P}(A \cup B)$, and similarly one has $\mathcal{P}B \subseteq \mathcal{P}(A \cup B)$. Now suppose $a \in \mathcal{P}A \cup \mathcal{P}B$. Then $a \in \mathcal{P}A$ or $a \in \mathcal{P}B$. In either case, $a \in \mathcal{P}(A \cup B)$ by the above. This shows $\mathcal{P}A \cup \mathcal{P}B \subseteq \mathcal{P}(A \cup B)$, as desired.

The converse inclusion does not always hold. For example, let A be the set of even numbers and B be the set of odd numbers. Then $\{2, 3, 4\} \in \mathcal{P}(A \cup B)$, but $\{2, 3, 4\} \notin \mathcal{P}A \cup \mathcal{P}B$.

The equality $\mathcal{P}(A \cup B) = \mathcal{P}A \cup \mathcal{P}B$ holds if and only if $B \subseteq A$ or $A \subseteq B$. To show the "if" part is easy enough; for the "only if" part, suppose that $\mathcal{P}(A \cup B) = \mathcal{P}A \cup \mathcal{P}B$. Since $A \cup B \in \mathcal{P}(A \cup B)$, it follows that $A \cup B \in \mathcal{P}A \cup \mathcal{P}B$, and thus $A \cup B \in \mathcal{P}A$ or $A \cup B \in \mathcal{P}B$. In the first case $B \subseteq A$, and in the second case $A \subseteq B$.

Problem 2.8 Suppose B was a set such that $\forall x\,(\{x\} \in B)$. Then for any set x, one would have $x \in \{x\} \in B$, and thus $x \in \bigcup B$. Thus $\bigcup B$ would have everything as a member, contradicting Theorem 2A.

Problem 2.9 Let a be any nonempty set, and let $B = \{a\}$. Clearly $a \in B$. We have $\emptyset \subseteq a$, and thus $\emptyset \in \mathcal{P}a$, but not $\emptyset \in B$. Thus, $\mathcal{P}a \not\subseteq B$, and hence $\mathcal{P}a \notin \mathcal{P}B$.