MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999

Answers to Problem Set 8

Problem 7.11 (a) Let $<'$ be the following ordering on the integers $\mathbb{Z}$:

$$0 <' 1 <' 2 <' \ldots <' -1 <' -2 <' -3 <' \ldots$$

Clearly, this is a linear ordering. To see that it is a well-ordering, consider any non-empty subset of $\mathbb{Z}$. Case 1: If A contains some non-negative number, then the set $B = \{n \in \omega \mid n \in A\}$ is non-empty and has a least element n_0 by the well-ordering property of ω (for convenience, we regard ω as a subset of $\mathbb{Z}$). Then for any $x \in A$, either x is non-negative and thus in B, in which case $n_0 \leqslant' x$ by leastness of n_0, or else x is negative, and thus $n_0 <' x$ by definition of $<'$. So n_0 is least in A.

Case 2: If A contains only negative numbers, then let $B' = \{m \in \omega \mid -m \in A\}$. The set B' is non-empty, and thus has a least element m_0 by the well-ordering property of ω. Then for any $x \in A$, we have $x = -m$ for some $m \in B'$, and thus $m_0 \leqslant m$, which implies $-m_0 \leqslant' -m = x$ by definition of $<'$. It follows that $-m_0$ is least in A. In any case, A has a least element, and thus $<'$ is a well-order.

(b) One has $E(3) = 3$, $E(-1) = \omega$, and $E(-2) = \omega^+$.

Problem 7.14 We use the non-strict version of the definition of a partial order: First, notice that if $a \leqslant b$, then for all $x \in A$, we have $x \in F(a) \Rightarrow x \leqslant a \Rightarrow x \leqslant b \Rightarrow x \in F(b)$ by transitivity, and thus $F(a) \subseteq F(b)$. Conversely, assume that $F(a) \subseteq F(b)$, for $a, b \in A$. Since $a \in F(a)$, we have $a \in F(b)$, and thus $a \leqslant b$, by definition of F. So for $a, b \in A$, we have $a \leqslant b$ iff $F(a) \subseteq F(b)$. It remains to be shown that F is one-to-one and onto. So suppose that $F(a) = F(b)$, then $F(a) \subseteq F(b)$ and $F(b) \subseteq F(a)$, thus $a \leqslant b$ and $b \leqslant a$ by what we have just shown, and thus $a = b$ by antisymmetry. So F is one-to-one. Also, F is clearly onto S since $S = \operatorname{ran} F$. It follows that F is an isomorphism from $\langle A, \leqslant\rangle$ onto $\langle S, \subseteq\rangle$.

Problem 7.16 Recall that for any sets, $x \in y^+$ iff $x \in\!\!\!\!= y$. This follows immediately from the definition of $y^+ = y \cup \{y\}$. Assume α and β are ordinal numbers with $\alpha \in \beta$. Then $\beta \notin\!\!\!\!= \alpha$ by trichotomy, and thus $\beta \notin \alpha^+$ by the above remark. But α^+ is an ordinal by Lemma 12(2), so by trichotomy again, it follows that $\alpha^+ \in\!\!\!\!= \beta$. This implies, again by the above remark, $\alpha^+ \in \beta^+$. Finally, if $\alpha \neq \beta$, then $\alpha \in \beta$ or $\beta \in \alpha$ by trichotomy. In the first case, $\alpha^+ \in \beta^+$, and in the second case, $\beta^+ \in \alpha^+$. In either case, $\alpha^+ \neq \beta^+$.

If we use the regularity axiom, then we can prove that for any two sets x and y (not necessarily ordinals), if $x \neq y$ then $x^+ \neq y^+$. Suppose to the contrary that $x^+ = y^+$. Then $x \in x^+ = y^+$ and $y \in y^+ = x^+$. So by our first remark, $x \in\!\!\!\!= y$ and $y \in\!\!\!\!= x$. But $x \neq y$, and thus $x \in y$ and $y \in x$, contradicting regularity.

Problem 7.17 We first show the following claim: If α, β are ordinals, and if there exists a one-to-one, order-preserving map $f : \beta \to \alpha$, then $\beta \in\!\!\!\!= \alpha$. Proof: We first prove, as in Problem 7.5, that for all $x \in \beta$, $x \in\!\!\!\!= f(x)$. For otherwise there would be a least $x \in \beta$ such that $x \notin\!\!\!\!= f(x)$. But $x \in \alpha$ and $f(x) \in \beta$ are ordinals, so by trichotomy, $f(x) \in x$. Since f is order-preserving, we have $f(f(x)) \in f(x)$, contradicting the leastness of x. This shows that for all $x \in \beta$, $x \in\!\!\!\!= f(x) \in \alpha$, and thus, by transitivity of α, $x \in \alpha$. In particular, it follows that $\beta \subseteq \alpha$, and by Corollary 8, $\beta \in\!\!\!\!= \alpha$.

Now Problem 7.17 follows easily: Let $E_A : A \to \alpha$ and $E_B : B \to \beta$ be the respective isomorphisms from $\langle A, <\rangle$ and $\langle B, <\rangle$ onto their $\in$-images. Then $E_A \circ {E_B}^{-1} : \beta \to \alpha$ is one-to-one and order-preserving, and thus $\beta \in\!\!\!\!= \alpha$ by our first claim.

Problem 7.18 Recall from Lemma 12(3) that if S is a set of ordinals, then $\bigcup S$ is an ordinal. We distinguish two cases. Case 1: $\bigcup S \in S$. In this case, for any $\alpha \in S$, we have $\alpha \subseteq \bigcup S$ (by definition of union), and thus $\alpha \in\!\!\!\!= \bigcup S$ (by Corollary 8). It follows that $\bigcup S$ is the greatest element of S.

Case 2: $\bigcup S \notin S$. We claim that S has no greatest element in this case. Assume, to the contrary, that α was a greatest element of S. Then for any $x \in \alpha$, we would have $x \in \bigcup S$ by definition of union. Conversely, for any $x \in \bigcup S$, we would have $x \in y \in S$ for some $y \in S$, but $y \in\!\!\!\!= \alpha$, and thus $x \in \alpha$ by transitivity of α. It follows that $\alpha = \bigcup S$, contradicting $\bigcup S \notin S$. Thus, S has no greatest element.

We further claim that in Case 2, $\bigcup S$ is not the successor of another ordinal. Suppose, to the contrary, that $\bigcup S = \beta^+$. Then $\beta \in \bigcup S$, and thus $\beta \in \alpha \in S$ for some ordinal α, hence $\bigcup S = \beta^+ \in\!\!\!\!= \alpha$. On the other hand, since $\alpha \in S$, we have $\alpha \subseteq \bigcup S$ by definition of union, and hence $\alpha \in\!\!\!\!= \bigcup S$, since $\bigcup S$ is an ordinal. It follows that $\bigcup S = \alpha \in S$, contradicting $\bigcup S \notin S$. Therefore, $\bigcup S$ is not the successor of an ordinal.