Math 4680, Topics in Logic and Computation, Winter 2012

Answers to Homework 3

Problem 2.1 #2 "There is no greatest interesting number."

Problem 2.2 #1 (a) "$\Rightarrow$": Suppose that $\Gamma;\alpha \models \varphi$. We want to show that $\Gamma \models (\alpha \to \varphi)$. To that end, consider any structure $\mathfrak{A}$ and any interpretation s, and assume $\models_{\mathfrak{A}} \Gamma[s]$. We must prove $\models_{\mathfrak{A}} (\alpha \to \varphi)[s]$. There are two cases: Case 1: $\models_{\mathfrak{A}} \alpha[s]$. In this case, using the assumption $\Gamma;\alpha \models \varphi$, it follows that $\models_{\mathfrak{A}} \varphi[s]$, and therefore also $\models_{\mathfrak{A}} (\alpha \to \varphi)[s]$ by definition of the interpretation of "$\to$" in a structure. Case 2: $\not\models_{\mathfrak{A}} \alpha[s]$. In this case, $\models_{\mathfrak{A}} (\alpha \to \varphi)[s]$ by definition of the interpretation of "$\to$" in a structure. In both cases, we are done.

"$\Leftarrow$": Suppose that $\Gamma \models (\alpha \to \varphi)$. We want to show that $\Gamma;\alpha \models \varphi$. So consider any structure $\mathfrak{A}$ and any interpretation s, and assume $\models_{\mathfrak{A}} \Gamma[s]$ and $\models_{\mathfrak{A}} \alpha[s]$. We must prove $\models_{\mathfrak{A}} \varphi[s]$. From the assumption $\Gamma \models (\alpha \to \varphi)$, it follows that $\models_{\mathfrak{A}} (\alpha \to \varphi)[s]$, and with $\models_{\mathfrak{A}} \alpha[s]$, it follows that $\models_{\mathfrak{A}} \varphi[s]$, as desired.

(b) "$\Rightarrow$": Suppose that $\varphi \models\!\dashv \psi$. We want to show that $\models (\varphi \leftrightarrow \psi)$. So consider any structure $\mathfrak{A}$ and any interpretation s. There are two cases: Case 1: $\models_{\mathfrak{A}} \varphi[s]$ and $\models_{\mathfrak{A}} \psi[s]$, in which case $\models_{\mathfrak{A}} (\varphi \leftrightarrow \psi)[s]$. Case 2: $\not\models_{\mathfrak{A}} \varphi[s]$ and $\not\models_{\mathfrak{A}} \psi[s]$, in which case $\models_{\mathfrak{A}} (\varphi \leftrightarrow \psi)[s]$. In either case, we are done.

"$\Leftarrow$": Suppose that $\models (\varphi \leftrightarrow \psi)$. We want to show that $\varphi \models\!\dashv \psi$. So consider any structure $\mathfrak{A}$ and any interpretation s. We must show that $\models_{\mathfrak{A}} \varphi[s]$ iff $\models_{\mathfrak{A}} \psi[s]$. By assumption, $\models (\varphi \leftrightarrow \psi)$, so either $\models_{\mathfrak{A}} \varphi[s]$ and $\models_{\mathfrak{A}} \psi[s]$, in which case we are done, or $\not\models_{\mathfrak{A}} \varphi[s]$ and $\not\models_{\mathfrak{A}} \psi[s]$, in which case we are also done.

Problem 2.2 #9 (a) $\exists x \exists y (x \neq y \wedge \forall z. z = x \vee z = y)$.

(b) $\forall x \exists y (P(x,y) \wedge \forall z (P(x,z) \to z = y))$.

(c) $\forall x \exists y (P(x,y) \wedge \forall z (P(x,z) \to z = y)) \wedge \forall y \exists x. P(x,y)$.

Problem 2.2 #14 (a) The only susets of the real line that are definable in $(\mathbb{R};<)$ are $\emptyset$ and $\mathbb{R}$.

Proof: suppose that A is some definable subset, i.e., there is some formula φ such that $x \in A$ iff $\models_{\mathbb{R}} \varphi(x)$. Suppose that A is not $\emptyset$ or $\mathbb{R}$. Then there are $x, y \in \mathbb{R}$ such that $x \in A$ and $y \notin A$. Let $f : \mathbb{R} \to \mathbb{R}$ be some monotone function such that $f(x) = y$ (Such a function always exists, for example, the function $f(z) = z + y - x$). Since f is an automorphism, we have by the homomorphism theorem that $\models_{\mathbb{R}} \varphi(x)$ iff $\models_{\mathbb{R}} \varphi(f(x))$ iff $\models_{\mathbb{R}} \varphi(y)$, so $x \in A$ iff $y \in A$, a contradiction. It follows that A is $\emptyset$ or $\mathbb{R}$.

(b) Consider the following subsets of $\mathbb{R} \times \mathbb{R}$:

$$\begin{aligned} X &= \{(x,y) \mid x < y\} \\ Y &= \{(x,y) \mid x = y\} \\ Z &= \{(x,y) \mid x > y\} \end{aligned}$$

Clearly, X, Y, and Z are definable (by the formulas $x < y$, $x = y$, and $x > y$, respectively; or if equality is not in the language, we can write $\neg x < y \wedge \neg y < x$ instead of $x = y$). We get the following 8 definable subsets:

$$\emptyset, \quad X, \quad Y, \quad Z, \quad X \cup Y, \quad X \cup Z, \quad Y \cup Z, \quad X \cup Y \cup Z = \mathbb{R} \times \mathbb{R}.$$

And these are in fact the only 8 definable subsets. Proof: Let $\varphi(x,y)$ be any formula defining a subset $A \subseteq \mathbb{R} \times \mathbb{R}$. Consider $(x,y) \in X$ and $(x',y') \in X$. We claim that $\varphi(x,y)$ iff $\varphi(x',y')$. Indeed, because we have $x < y$ and $x' < y'$, we can find a monotone function f such that $f(x) = x'$ and $f(y) = y'$. By the homomorphism theorem, it follows that $\varphi(x,y)$ iff $\varphi(f(x), f(y)$ iff $\varphi(x',y')$. Therefore, A either contains all elements of X or no elements of X.

Similarly, we prove that A either contains all or no elements of Y, and A either contains all or no elements of Z. It follows that A is of one of the 8 forms above.