MATH 1500, THE CALCULUS, Fall 2013

Handout 1: Optimization problems
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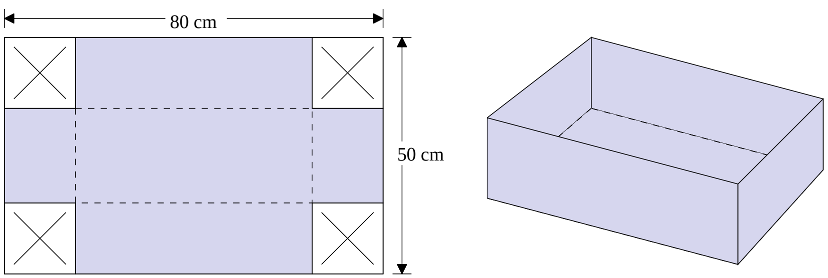
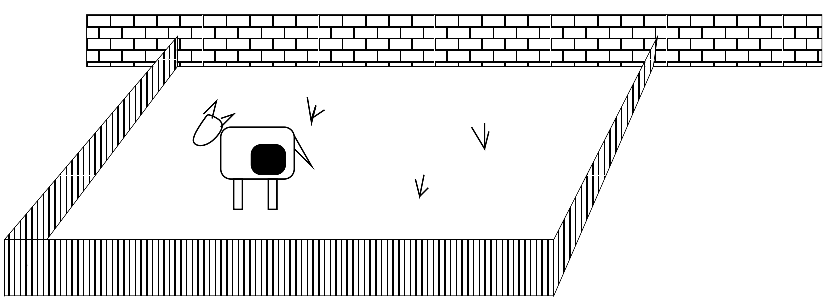
Problem 1. We have a piece of cardboard of size $50\,\mathrm{cm} \times 80\,\mathrm{cm}$. From this, we would like to construct a lid-less box by cutting away for squares and folding the sides up, as shown in the illustration:



Determine the size of the cutouts that maximize the volume of the box. What is the maximum volume?

Problem 2. A ship is travelling due north at a speed of $45\,\mathrm{km/h}$, passing the point P at 8am. A second ship is travelling due east at a speed of $15\,\mathrm{km/h}$, reaching the point P at 9am. At what time will the ships be closest to each other? What will be their distance at this moment?

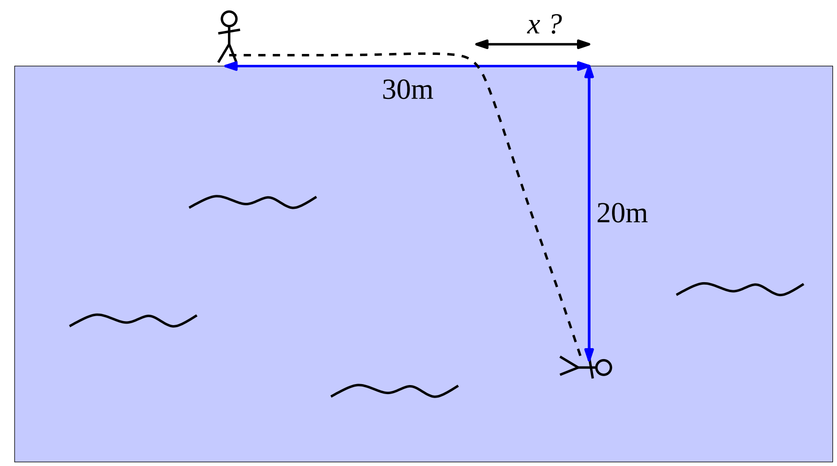
Problem 3. A farmer wishes to fence off a rectangular area. Since the area is along a wall, she only needs fence on three sides of the rectangle, as shown in the illustration. She has 100 meters of fence. What is the largest area she can fence off?



Problem 4. A square box is to be constructed with a bottom and four sides (but no top). What is the largest volume of a box that can be constructed with $12\,\mathrm{m}^2$ of material? What are the dimensions of the box?

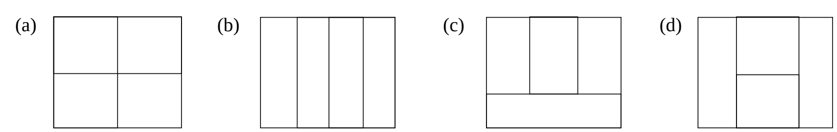
Note: unlike in Problem 1, the shape of the material does not matter here, only the amount of material actually used.

Problem 5. A life guard is sitting at the edge of a pool. A swimmer calls for help. The swimmer is located 30 m down the edge of the pool and 20 m into the water, as shown in the illustration. The life guard can run 5 times faster than he can swim. What is the path that he should take to get to the swimmer as quickly as possible?

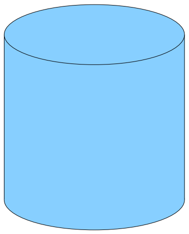


Problem 6. Find the points on the parabola $y = x^2$ that are closest to the point $(0, 2)$.

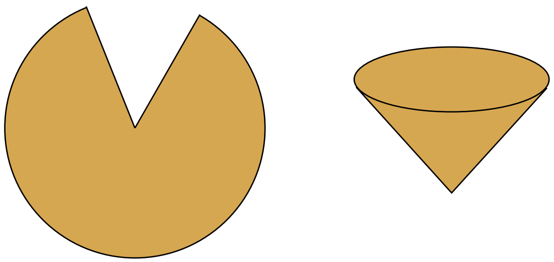
Problem 7. A farmer with 120 m of fencing would like to enclose a rectangular area divided into 4 pens of equal area. What is the maximum possible total area of the pens, if the arrangement is as in (a)? As in (b)? As in (c)? As in (d)? Which of the four arrangements is the best?



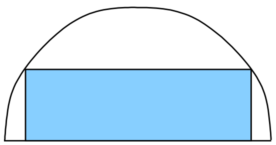
Problem 8. A cylindrical storage container is to be constructed to have a volume of 10 liters. What should the dimensions of the container be to minimize the amount of material used? The container has both a top and a bottom.



Problem 9. An ice cream cone is formed from a circular waffle of radius 10 cm, by cutting out a sector and joining the resulting straight edges together. What is the maximum volume of the resulting ice cream cone?



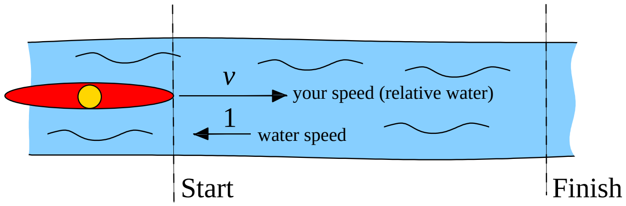
Problem 10. A rectangle is to be inscribed in a semicircle of radius 1, as shown in the illustration. What is the largest possible area of the rectangle? What are its dimensions?



Problem 11. A poster is to be printed on a piece of paper whose area is 1.25 square meters. The printer requires margins of 5 cm at the top and bottom, and 4 cm on the left and right. Find the dimensions of the poster that maximizes the size of the printed area.

Problem 12. You are kayaking upstream, trying to reach a distant destination. The speed of the water is 1 m/s. Your own speed, relative to the current, is v. To keep up your speed, the amount of energy you need to expend per second is proportional to v^3. How fast should you go to minimize the total energy required for the journey?

Note: Since your speed relative to the water is v, your speed relative to the ground is $v - 1$. The amount of time it takes you to go distance D is $D/(v-1)$. Therefore, the total energy required to go distance D is $Cv^3D/(v-1)$, where C is a proportionality constant.



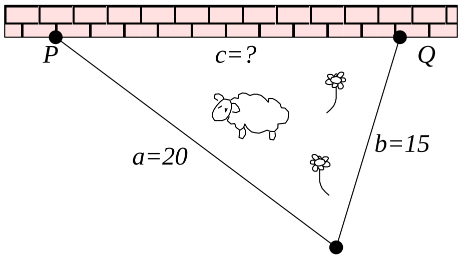
Remark: The result predicted by this exercise has been experimentally verified in migratory fish. The fish swim against the current at a speed that minimizes the required energy per distance traveled.

Problem 13. A farmer has two pieces of straight wooden fence, of length 15 m and 20 m, respectively. He wants to fence off a triangular area against a wall, as shown in the illustration. How far apart should the points P and Q be to maximize the area?

Note: By *Heron's formula*, the area A of a triangle with sides a, b, and c satisfies

$$16A^2 = (a+b+c)(a+b-c)(a-b+c)(-a+b+c).$$

Hint: instead of minimizing the area A, the farmer may equivalently minimize the square of the area, A^2. This will simplify your calculations!



Source: several problems adapted from Stewart's "Single Variable Calculus"