Orthogonality Logic

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Orthogonal Subcategory Problem and Orthogonality Logic

category $\mathcal{A}$, $\mathcal{H} \subseteq \text{Mor}(\mathcal{A})$

$\mathcal{H}^{\perp} := \text{full subcategory of } \mathcal{A}\text{-objects orthogonal to } \mathcal{H}$
Orthogonal Subcategory Problem and Orthogonality Logic

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\begin{array}{c}
A \xrightarrow{r_A} \overline{A}
\end{array}
\]

the construction of the reflection involves categorical "rules" (composition, limits, colimits, factorization, ...)

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Question: When are these "rules" part of a sound and complete deduction system for orthogonality?
Find a Deduction System of \emph{RULES} such that

\[ h \in \big( \mathcal{H}^\perp \big)^\perp \iff h \text{ is deducible from } \mathcal{H} \text{ by successively applying the } \text{RULES} \]
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\[ \mathcal{H} \models h \iff \mathcal{H} \vdash h \]
Orthogonal Subcategory Problem and Orthogonality Logic

Find a Deduction System of *RULES* such that

\[ h \in \left( \mathcal{H} \perp \right) \perp \iff h \text{ is deducible from } \mathcal{H} \text{ by successively applying the } *RULES*

\[ \mathcal{H} \models h \iff \mathcal{H} \vdash h \]

\[ \mathcal{H} \models h := (A \perp \mathcal{H} \Rightarrow A \perp h), \text{ for all objects } A \]

\[ \mathcal{H} \vdash h := \text{there is a formal proof of } h \text{ from } \mathcal{H} \text{ by using the Deduction System} \]
The Finitary Case: Sentences versus Morphisms

\[ e \equiv (u = v) \]
\[ u \text{ and } v \text{ terms in } X \]

\[ q_e : FX \to FX/ \sim_e \]

algebras satisfying
\[ \mathbb{E} = \{e_i, i \in I\}, \; e_i \equiv (u_i = v_i) \]

algebras orthogonal to
\[ \mathbb{E}' = \{q_{e_i}, i \in I\} \]
The Finitary Case: Sentences versus Morphisms

\[ e \equiv (u = v) \quad \quad q_e : FX \to FX/ \sim_e \]

\( u \) and \( v \) terms in \( X \)

algebras satisfying \( \mathbb{E} = \{e_i, i \in I\}, e_i \equiv (u_i = v_i) \)  
algebras orthogonal to \( \mathbb{E}' = \{qe_i, i \in I\} \)

Analogously for implications and regular sentences
### The Finitary Case: Sentences versus Morphisms

<table>
<thead>
<tr>
<th>$A$ satisfies</th>
<th>$A$ is orthogonal to</th>
</tr>
</thead>
<tbody>
<tr>
<td>equations</td>
<td>epimorphisms with projective domain (orthogonality=inject.)</td>
</tr>
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<td>$\forall x E(x)$</td>
<td></td>
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<td>implications</td>
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<tr>
<td>$\forall x (E(x) \rightarrow F(x))$</td>
<td></td>
</tr>
<tr>
<td>limit sentences</td>
<td>morphisms</td>
</tr>
<tr>
<td>$\forall x (E(x) \rightarrow \exists! y F(x, y))$</td>
<td></td>
</tr>
</tbody>
</table>

$E(x)$ and $F(x)$ involving a finite number of variables and equations, finitary morphisms, i.e., with finitely presentable domain and codomain.
A sound and complete deduction system for finitary epimorphisms with projective domains

A sound and complete deduction system for finitary epimorphisms
Finitary Logic

\( \mathcal{A} \) a finitely presentable category

- Formulas: finitary morphisms, i.e., morphisms of \( \mathcal{A}_{fp} \)
- Formal proofs have only a finite number of steps
If $\mathcal{F}$ is a set of finitary morphisms admitting a left calculus of fractions (in $\mathcal{A}_{fp}$) then $\mathcal{F}^\perp$ is reflective in $\mathcal{A}$.

sound rules

IDENTITY

\[ \text{id}_A \]

COMPOSITION

\[ \frac{h_1 \ h_2}{h_2 \cdot h_1} \]

PUSHOUT

\[ \frac{h}{h'} \]

if

\[ \begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array} \]

COEQUALIZER

\[ \frac{h}{h'} \]

if

\[ \frac{h}{f} \]

\[ g 
\]

\[ h' \]

\[ f \cdot h = g \cdot h \]

\[ h' = \text{coeq}(f, g) \]
Soundness of **COEQUALIZER**

\[ \frac{h}{h'} \]

\[ h \xrightarrow{f} \xrightarrow{g} h' \]
Soundness of COEQUALIZER

\[ \frac{h}{h'} \]

Diagram:

\[ h \xrightarrow{f} g \xrightarrow{h'} \]

\[ x \downarrow \quad X \]
Soundness of COEQUALIZER

\[ \frac{h}{h'} \]

\[
\begin{array}{c}
\text{Soundness of COEQUALIZER } \frac{h}{h'} \\
\text{(xf)h = (xg)h} \Rightarrow xf = xg
\end{array}
\]
Soundness of COEQUALIZER

\[
\begin{array}{c}
h \quad \xrightarrow{f} \quad h' \\
g \quad \downarrow \quad \quad \quad \downarrow \\
X \\
\end{array}
\]

\[
(xf)h = (xg)h \Rightarrow xf = xg
\]
CANCELLATION is not sound

\[ \{0\} \xrightarrow{f} \{0, 1\} \xrightarrow{g} \{0\} \]

\[ g \cdot f = \text{id}_{\{0\}} \not\models f \]

because \( \{0, 1\} \models \text{id}_{\{0\}} \) but \( \{0, 1\} \not\models f \)
\( \nabla\text{-CANCELLATION} \)

\[
\begin{array}{c}
\frac{f \cdot h}{\nabla h} \\
\hline
\end{array}
\]

\[ A \xrightarrow{h} B \]

\[ B \xrightarrow{v} C \]

\[ B \xleftarrow{1_B} \]

\[ B \xrightarrow{1_B} \]

\[ h \]

\[ u \]

\[ \nabla h \]
\( \nabla \)-CANCELLATION

\[
\frac{f \cdot h}{\nabla h} \quad h
\]

is sound:
\( \nabla \text{-CANCELLATION} \)

\[
\frac{f \cdot h}{\nabla h} \quad h
\]

is sound:

\[
\begin{array}{c}
A \xrightarrow{h} B \\
\downarrow h \quad \downarrow u \\
B \xrightarrow{v} C \\
\downarrow 1_B \\
B \xrightarrow{k} X
\end{array}
\]
∇-CANCELLATION

\[
\frac{f \cdot h \quad \nabla h}{h}
\]

is sound:

\[
\begin{array}{c}
A \\ h
\end{array} 
\xrightarrow{h} 
\begin{array}{c}
B \\ f
\end{array} 
\xrightarrow{f} 
\begin{array}{c}
C \\ v
\end{array} 
\xrightarrow{v} 
\begin{array}{c}
B \\ u
\end{array} 
\xrightarrow{u} 
\begin{array}{c}
X
\end{array}
\]
\[ \nabla \text{-CANCELLATION} \]

\[
\frac{f \cdot h \quad \nabla h}{h}
\]

is sound:

\[
\begin{array}{c}
A \\ h \\ \downarrow \\
B \\
\downarrow h \\
B \\
\downarrow \downarrow \\
B \\
\downarrow v \\
C \\
\downarrow u \\
B \\
\downarrow 1_B \\
B \\
\downarrow \nabla h \\
B \\
\downarrow q \\
X \\
\end{array}
\]
is sound:
is sound:

\[ p = t' \cdot \nabla_h u = t' \cdot \nabla_h v = q \]
Finitary Orthogonality Deduction System

IDENTITY
\[ \text{id}_A \]

COMPOSITION
\[ \frac{h_1 \ h_2}{h_2 \cdot h_1} \]

PUSHOUT
\[ \frac{h}{h'} \quad \text{if} \quad \frac{h}{h'} \]

COEQUALIZER
\[ \frac{h}{h'} \quad \text{if} \quad \frac{h}{h'} \]

\[ fh = gh, \ h' = \text{coeq}(f, g) \]

∇-CANCELLATION
\[ \frac{f \cdot h}{\nabla_h} \]

– p. 13/25
The Finitary Orthogonality Deduction System is sound and complete, that is,

\[ \mathcal{H} \models h \iff \mathcal{H} \vdash h \]
Finitary Orthogonality Deduction System

**IDENTITY**

\[ \text{id}_A \]

**COMPOSITION**

\[ h_2 \cdot h_1 \]

**PUSHOUT**

\[ h \quad h' \]

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\[ f \cdot h \quad \nabla_h \]
IDENTITY
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\[ h \quad h' \]

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\[ h \quad h' \]

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\[ f \cdot h \quad \nabla_h \]
IDENTITY

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TRANSFINITE COMPOSITION

\[ h_i, \ i \in \alpha \]

\[ h \]

PUSHOUT

\[ h \]

\[ h' \]

COEQUALIZER

\[ h \]

\[ h' \]

\[ f \cdot h \]

\[ \nabla_h \]

\[ h \]

\[ h' \]

\[ h_1 \]

\[ h_2 \]

\[ \cdots \]

\[ h \]

\[ h' \]

\[ f \]

\[ g \]
Finitary Orthogonality Deduction System

IDENTITY

TRANSFINITE COMPOSITION

PUSHOUT

COEQUALIZER

∇-CANCELLATION
Orthogonality Deduction System

TRANSFINITE COMPOSITION
\[ h_i, \; i \in \alpha \]  
\[ \underbrace{h} \]
\[ h_1 \rightarrow h_2 \rightarrow \ldots \]

PUSHOUT
\[ h \]
\[ \underbrace{h'} \]
\[ h \rightarrow h' \]

COEQUAIZER
\[ h \]
\[ \underbrace{h'} \]
\[ h \rightarrow f \rightarrow h' \]

∇-CANCELLATION
\[ f \cdot h \]
\[ \nabla h \]
\[ \underbrace{h} \]
The Orthogonality Deduction System is sound and complete. That is,

\[ \mathcal{H} \models h \iff \mathcal{H} \vdash h \]
Incompleteness Example: a cocomplete category where the Orthogonality Logic is not complete

\[ \text{CPO}_\perp(1) \]

Objects: \((X, \leq, \alpha)\), where \((X, \leq)\) is a CPO with a least element, and \(\alpha : X \to X\)

Morphisms: continuous maps preserving the least element and the unary operation
\[ x < \alpha(x) \]
\{h_1, h_2\} \models h \text{ but } \{h_1, h_2\} \not\models h
$f : A \rightarrow \text{Ord}$ is a coloring of $A$, that is:

- $f$ is continuous, $f(\bot) = 0$
- and $f(\alpha(x)) = f(x) + 1$

$\mathcal{K}(A, K) = \{\text{colorings of } A\}$
$f : A \to Ord$ is a coloring of $A$, that is:

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$$\mathcal{K}(A, K) = \{\text{colorings of } A\}$$

In $\text{CPO}_\bot(1)$, $\{h_1, h_2\} \models h$

But in $\mathcal{K}$, $K$ is orthogonal to $\{h_1, h_2\}$ but is NOT orthogonal to $h$.

Then: In $\text{CPO}_\bot(1)$, $\{h_1, h_2\} \models h$, but $\{h_1, h_2\} \not\vdash h$. 

- p. 23/25
In the Orthogonality Deduction System, for sets $\mathcal{H}$,

$$\mathcal{H} \models h \iff \mathcal{H} \vdash h$$
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Question: What about the completeness when we admit a proper class of morphisms $\mathcal{H}$ as premises?
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$\mathcal{H} \models h$ iff $\mathcal{H} \vdash h$

Question: What about the completeness when we admit a proper class of morphisms $\mathcal{H}$ as premises?

Special classes:

Classes of epimorphisms: Yes
Classes where just a set of morphisms are not epimorphisms: ??
The completeness for classes of the Orthogonality Logic (in locally presentable categories) is equivalent to the Vopěnka’s Principle.

existence of
huge cardinals
\[\Downarrow\]
Vopěnka’s Principle \(:=\) \(Ord\) has no full embedding into a loc. pres. cat.
\[\Downarrow\]
existence of
measurable cardinals