Proof nets and semi-$\star$-autonomous categories

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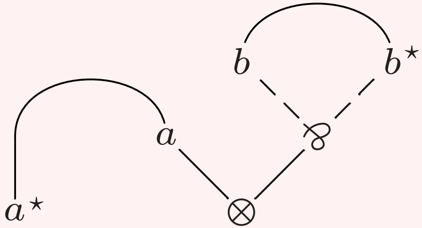
The problem

- Categories are a natural semantics for logic
- Girard's proof nets for MLL^- describe a free category (as do certain proof nets for additive linear logic)
- How to capture the nets (and subnets!) with a single conclusion?



- Traditionally as maps $I \to A$ (where I is the unit to the tensor)
- Adding I introduces formulae such as $\bot = I^\star$ and $I \parr (\bot \otimes \bot)$

Overview

- Proof nets for MLL^-
- The virtual unit
- Semi-$\star$-autonomous categories
- Related work
- Wire diagrams
- Proving the main theorem

Proof nets for MLL^-

MLL^{-}

Multiplicative linear logic without units

$$A \ := \ a \ \mid \ a^\star \ \mid \ A \otimes A \ \mid \ A \parr A$$

general duality by DeMorgan

$$a^{\star\star} = a \qquad (A \otimes B)^\star = A^\star \parr B^\star \qquad (A \parr B)^\star = A^\star \otimes B^\star$$

MLL^-

Multiplicative linear logic without units

$$A \;:=\; a \mid a^\star \mid A \otimes A \mid A \parr A$$

general duality by DeMorgan

$$a^{\star\star} = a \qquad (A \otimes B)^\star = A^\star \parr B^\star \qquad (A \parr B)^\star = A^\star \otimes B^\star$$

Formulae are annotated with vertices to serve as graphical objects:

$$A_U \;:=\; \underbrace{a_u \mid a^\star_u}_{U=\{u\}} \;\mid\; \underbrace{B_V \otimes_u C_W \mid B_V \parr_u C_W}_{U=\{u\}\uplus V \uplus W}$$

A sequent is a multiset Γ_V of disjointly annotated formulae.

Proof nets

A pre-proof net is a sequent Γ_V together with a linking $\mathcal{L}$: a partitioning of the atomic vertices in Γ_V into dual pairs.

$$\mathcal{L} \blacktriangleright [\Gamma_V]$$

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A switching graph for a pre-proof net is an undirected graph

$$(V, \mathcal{L} \cup S)$$

where S contains one edge $\langle u, v \rangle$ for every par-vertex ($\parr_u$), and all edges $\langle u, v \rangle$ for every tensor-vertex ($\otimes_u$), where v is a child of u.

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A proof net is a pre-proof net for which every switching graph is acyclic and connected.

Sequent proofs construct proof nets

A pre-net is a proof net if and only if it is constructed by the following sequent calculus. [Danos & Regnier]

$$\frac{}{\{\langle v, w\rangle\} \blacktriangleright [a_v, a^\star_w]}\ \mathbf{AX} \qquad \frac{\mathcal{L} \blacktriangleright [\Gamma_X, A_V, B_W]}{\mathcal{L} \blacktriangleright [\Gamma_X, A_V \mathbin{⅋_u} B_W]}\ ⅋\mathbf{R}$$

$$\frac{\mathcal{L} \blacktriangleright [\Gamma_X, A_V] \qquad \mathcal{K} \blacktriangleright [\Delta_Y, B_W]}{\mathcal{L} \cup \mathcal{K} \blacktriangleright [\Gamma_X, \Delta_Y, A_V \otimes_u B_W]}\ \otimes\mathbf{R}$$

Composition

Composition of proof nets is via the cut rule:

$$\frac{\mathcal{L} \blacktriangleright [\Gamma_U, A_V] \qquad \mathcal{K} \blacktriangleright [A^\star_V, \Delta_W]}{\mathcal{L};\mathcal{K} \blacktriangleright [\Gamma_U, \Delta_W]} \textbf{ Cut}$$

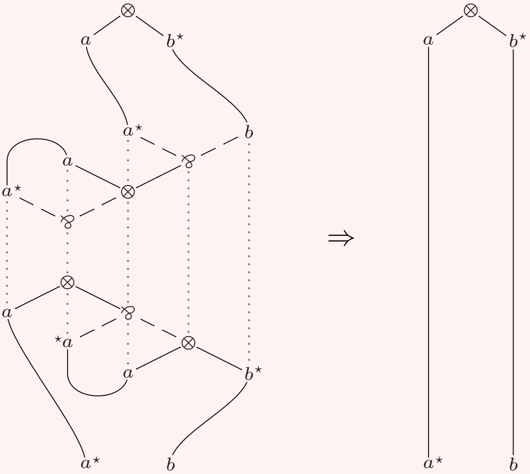
where $\mathcal{L};\mathcal{K}$ contains the link $\langle u, w \rangle$ precisely when there is a path

$$\langle u, v_1 \rangle, \langle v_1, v_2 \rangle, \ldots, \langle v_n, w \rangle$$

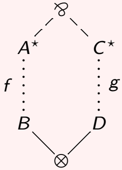
of links (alternately) from $\mathcal{L}$ and from $\mathcal{K}$

Composition is associative and has identities

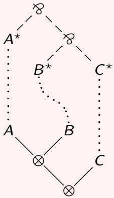
Composition (example)



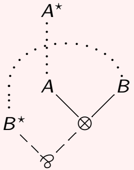
Categorical structure in proof nets



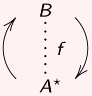
$f \otimes g$



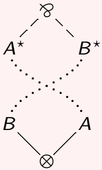
α (assoc.)



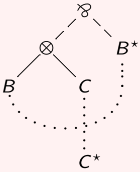
η (exp.)



$f^\star$



σ (symm.)



ϵ (eval.)

A $\star$-autonomous category without units

Definition

A tensor–dual category (TD category) $(\mathcal{C}, \otimes, ^\star)$ is a category $\mathcal{C}$ with

- a tensor bifunctor $(-\otimes-)$,
- a dualising functor $(-)^\star$, and
- the following natural isomorphisms,

$$\alpha\colon A \otimes (B \otimes C) \cong (A \otimes B) \otimes C \qquad \sigma\colon A \otimes B \cong B \otimes A$$

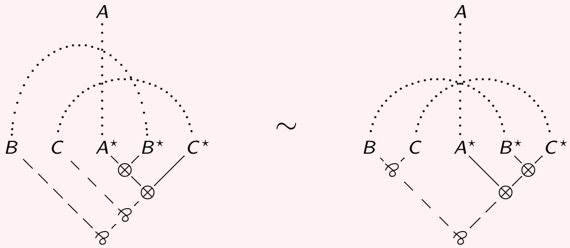
$$\partial\colon A \cong A^{\star\star} \qquad \Phi\colon \hom(A \otimes B, C^\star) \cong \hom(A, (B \otimes C)^\star)$$

satisfying the associativity pentagon, the symmetry hexagon, and

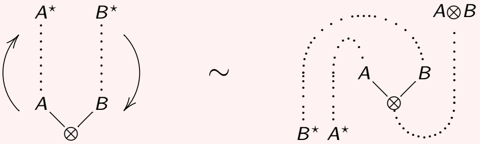
...

Coherence axioms for TD categories (I)

$$
\begin{array}{ccc}
 & \mathrm{hom}(A \otimes B, (C \otimes D)^\star) & \\
 \overset{\Phi}{\nearrow} & & \overset{\Phi}{\searrow} \\
\mathrm{hom}((A \otimes B) \otimes C, D^\star) & & \mathrm{hom}(A, (B \otimes (C \otimes D))^\star) \\
{\scriptstyle -\circ\alpha}\big\downarrow & \boldsymbol{\Phi\alpha} & \big\uparrow{\scriptstyle \alpha^\star\circ -} \\
\mathrm{hom}(A \otimes (B \otimes C), D^\star) & \xrightarrow{\quad\Phi\quad} & \mathrm{hom}(A, ((B \otimes C) \otimes D)^\star)
\end{array}
$$



Coherence axioms for TD categories (II)

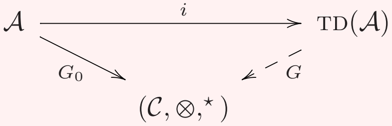


$$
\begin{array}{ccccc}
\hom(A \otimes B, C^\star) & \xrightarrow{(-)^\star} & \hom(C^{\star\star}, (A \otimes B)^\star) & \xrightarrow{-\circ\partial} & \hom(C, (A \otimes B)^\star) \\
\downarrow \Phi & & & & \uparrow \Phi \\
\hom(A, (B \otimes C)^\star) & & \boldsymbol{\Phi\sigma} & & \hom(C \otimes A, B^\star) \\
& \searrow \sigma^\star \circ - & & -\circ\sigma \nearrow & \\
& \hom(A, (C \otimes B)^\star) & \xrightarrow{\Phi^{-1}} & \hom(A \otimes C, B^\star) &
\end{array}
$$

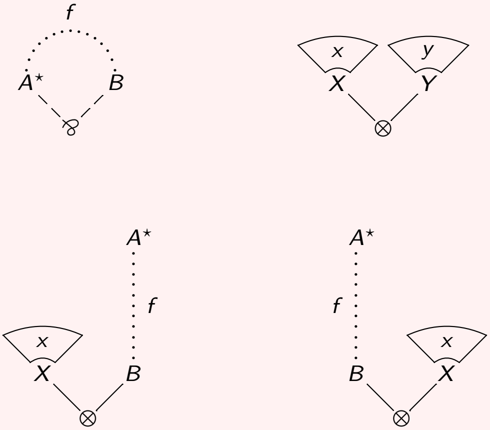
Proof nets and TD categories

Theorem

The subcategory of proof nets $\mathcal{L} \blacktriangleright [A^\star, B]$ *with* *no single-conclusion subnets* *is the* *free tensor–dual category* $\mathrm{TD}(\mathcal{A})$ *over the atoms* $\mathcal{A}$.

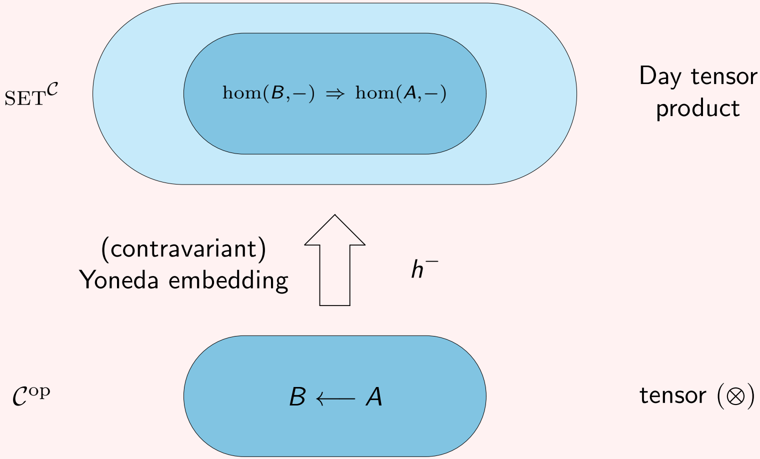


Missing nets



The virtual unit

Yoneda


$\text{SET}^{\mathcal{C}}$
$\hom(B,-) \Rightarrow \hom(A,-)$
Day tensor product
(contravariant) Yoneda embedding
h^-
$\mathcal{C}^{\text{op}}$
$B \longleftarrow A$
tensor $(\otimes)$

The virtual unit

Idea: find a virtual unit in $\textsc{set}^{\mathcal{C}}$ [Lamarche & Straßburger 2005]

$$\mathbb{I}\colon \mathcal{C} \to \textsc{set}$$

A proof net $\mathcal{L} \blacktriangleright [A]$ may be modelled by a natural transformation

$$\kappa\colon h^{A} \Rightarrow \mathbb{I} \qquad\qquad (\cong \;\; I \to A \quad \text{if} \quad \mathbb{I} = h^{I})$$

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$$\begin{array}{ccc} \hom(A,A) & \xrightarrow{\kappa_A} & \mathbb{I}A \\ {\scriptstyle f\circ -}\ \big\downarrow\ {\scriptstyle h^A(f)} & & \big\downarrow {\scriptstyle \mathbb{I}(f)} \\ \hom(A,B) & \xrightarrow[\kappa_B]{} & \mathbb{I}B \end{array} \qquad\qquad \begin{array}{ccc} id_A & \overset{\kappa_A}{\longmapsto} & \kappa_A(id_A) \\ {\scriptstyle f\circ -}\ \big\downarrow & & \big\downarrow {\scriptstyle \mathbb{I}(f)} \\ f & \underset{\kappa_B}{\longmapsto} & \kappa_B(f) \\ & & = \\ & & \mathbb{I}(f)(\kappa_A(id_A)) \end{array}$$

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Idea: find a virtual unit in $\mathrm{SET}^{\mathcal{C}}$ [Lamarche & Straßburger 2005]

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Aim: $\mathbb{I}A \;\cong$ the set of proof nets for A

The (left) virtual unit isomorphism

The internal hom-functor $H^B = (B^\star \otimes -)$ gives a 'tensor' in $\mathrm{SET}^{\mathcal{C}}$

$$h^{A\otimes B} \;=\; \mathrm{hom}(A \otimes B, -) \;\overset{\Phi}{\cong}\; \mathrm{hom}(A, B^\star \otimes -) \;=\; h^A \circ H^B$$

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Define an isomorphism λ to make $\mathbb{I}$ a left unit

$$\lambda : \mathbb{I} \circ H^- \cong h^- \qquad\qquad \lambda_A : \mathbb{I}(A^\star \parr -) \cong \hom(A, -)$$

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$$\begin{array}{ccc}
\mathbb{I}(A^\star \parr B) & \xrightarrow[\cong]{\lambda_{A,B}} & \mathrm{hom}(A, B) \\
{\scriptstyle \mathbb{I}(f^\star \parr g)} \downarrow & \mathbf{nat}(\boldsymbol{\lambda}) & \downarrow {\scriptstyle g \circ - \circ f} \\
\mathbb{I}(X^\star \parr Y) & \xrightarrow[\lambda_{X,Y}]{\cong} & \mathrm{hom}(X, Y)
\end{array}$$

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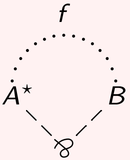
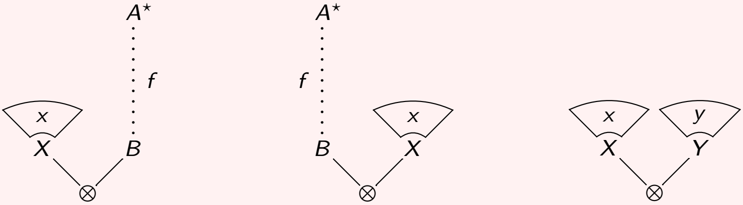


$\lambda^{-1}(f)$

The virtual tensors



$x \otimes f$ $\quad$ $f \otimes x$ $\quad$ $x \otimes y$

The virtual tensor in $\mathrm{SET}^{\mathcal{C}}$

...acts on morphisms as horizontal composition of two-cells.

$$\mathbb{I}(-)(x)\colon h^X \Rightarrow \mathbb{I} \qquad\qquad H^f\colon H^B \Rightarrow H^A$$

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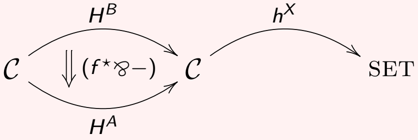

$$(f^\star \invamp id) \circ \quad - \quad \colon \mathsf{hom}(X, B^\star \invamp -) \Rightarrow \mathsf{hom}(X, A^\star \invamp -)$$

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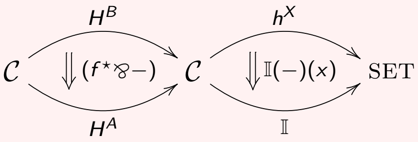

$$\mathbb{I}((f^\star \parr id) \circ \quad - \quad)(x) \quad : \ \mathrm{hom}(X, B^\star \parr -) \Rightarrow \mathbb{I}(A^\star \parr -)$$

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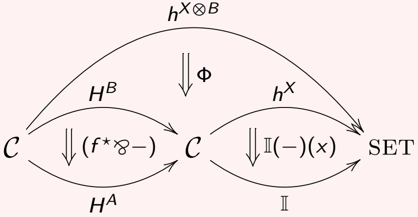

$$\mathbb{I}((f^\star \parr id) \circ \Phi(\,-\,))(x) \quad : \ \hom(X \otimes B, -) \Rightarrow \mathbb{I}(A^\star \parr -)$$

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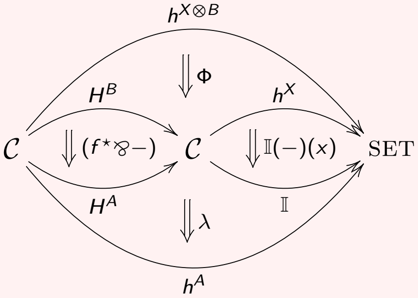

$$\lambda(\mathbb{I}((f^\star \otimes id) \circ \Phi(\,-\,))(x)) \;:\; \mathrm{hom}(X \otimes B, -) \Rightarrow \mathrm{hom}(A, -)$$

The virtual tensor in $\mathcal{C}$

Given the virtual tensor of x with f in $\textsc{Set}^{\mathcal{C}}$,

$$\lambda(\mathbb{I}((f^\star \invamp id) \circ \Phi(-))(x)) \;\;:\;\; \hom(X \otimes B, -) \Rightarrow \hom(A, -)$$

to obtain $x \otimes f$ in $\mathcal{C}$, apply this transformation to $id_{X \otimes B}$

$$x \otimes f \triangleq \lambda(\mathbb{I}((f^\star \invamp id) \circ \Phi(id))(x))$$

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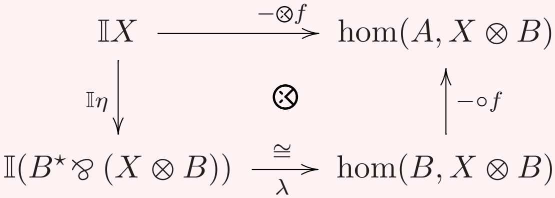
$$\begin{aligned} x \otimes f &\triangleq \lambda(\mathbb{I}((f^\star \invamp id) \circ \Phi(id))(x)) \\ &= \lambda(\mathbb{I}((f^\star \invamp id) \circ \eta)(x)) \\ &= \lambda(\mathbb{I}(\eta)(x)) \circ f \end{aligned}$$



The other two virtual tensors

$$f \otimes x \;\triangleq\; \sigma \circ (x \otimes f) \qquad : A \to B \otimes X$$

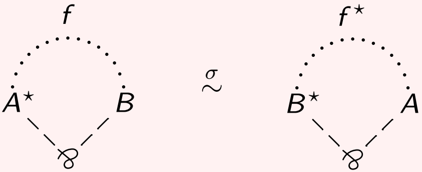
$$x \otimes y \;\triangleq\; \mathbb{I}(x \otimes id_Y)(y) \quad \in \mathbb{I}(X \otimes Y)$$

$$\begin{array}{ccc}
\mathbb{I}X \times \mathbb{I}Y & \xrightarrow{(-\otimes id)\times \mathbb{I}Y} & \hom(Y, X \otimes Y) \times \mathbb{I}Y \\
{\scriptstyle -\otimes-}\downarrow & \otimes & \downarrow{\scriptstyle \mathbb{I}\times\mathbb{I}Y} \\
\mathbb{I}(X \otimes Y) & \xleftarrow[\text{apply}]{} & \hom(\mathbb{I}Y, \mathbb{I}(X \otimes Y)) \times \mathbb{I}Y
\end{array}$$

Semi-$\star$-autonomous categories

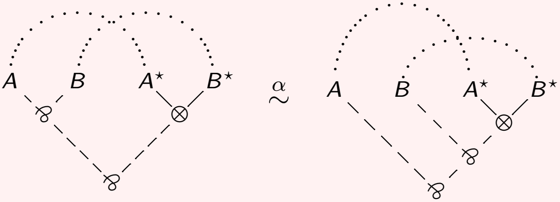
Coherence for the virtual tensor

$$
\begin{array}{ccc}
\mathbb{I}(A \parr B) & \xrightarrow{\mathbb{I}(\sigma^\star)} & \mathbb{I}(B \parr A) \\
\downarrow{\scriptstyle\lambda} & \boldsymbol{\lambda\sigma}^\star & \downarrow{\scriptstyle\lambda} \\
\hom(A^\star, B) & \xrightarrow[(-)^\star]{} & \hom(B^\star, A)
\end{array}
$$



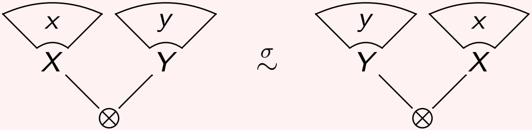
Coherence for the virtual tensor

$$
\begin{array}{ccc}
\mathbb{I}((A \parr B) \parr C)) & \xrightarrow{\mathbb{I}(\alpha^\star)} & \mathbb{I}(A \parr (B \parr C)) \\
\downarrow \lambda & \boldsymbol{\lambda\alpha^\star} & \downarrow \lambda \\
\hom(A^\star \otimes B^\star, C) & \xrightarrow[\Phi]{} & \hom(A^\star, B \parr C)
\end{array}
$$

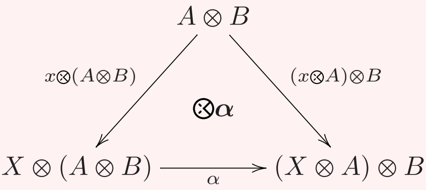


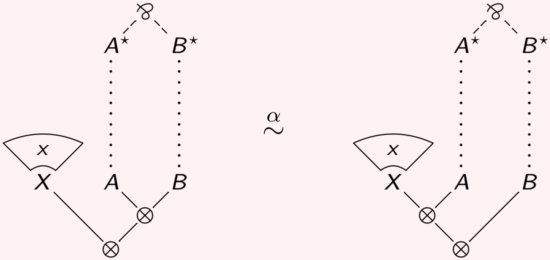
Coherence for the virtual tensor

$$
\begin{array}{ccc}
\mathbb{I}A \times \mathbb{I}B & \xrightarrow{\ \sigma\ } & \mathbb{I}B \times \mathbb{I}A \\
{\scriptstyle -\circledast-}\downarrow & \circledast\boldsymbol{\sigma} & \downarrow{\scriptstyle -\circledast-} \\
\mathbb{I}(A \otimes B) & \xrightarrow[\ \mathbb{I}\sigma\ ]{} & \mathbb{I}(B \otimes A)
\end{array}
$$



Coherence for the virtual tensor

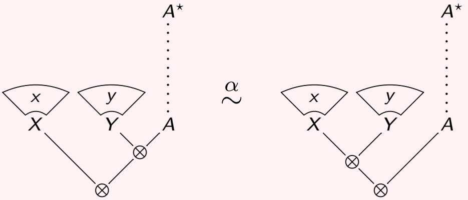




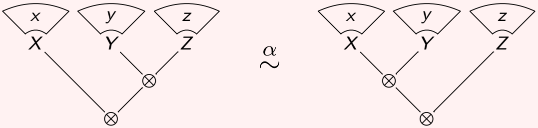
Coherence for the virtual tensor

From the latter axiom follow:

$\alpha \circ x \otimes (y \otimes A) = (x \otimes y) \otimes A$



$\mathbb{I}(\alpha)(\ x \otimes (y \otimes z)\ ) = (x \otimes y) \otimes z$



$+$ all (four) symmetric variants

Semi-$\star$-autonomous categories

Definition

A semi-$\star$-autonomous category (SSA category)

$$(\mathcal{C}, \otimes, ^\star, \mathbb{I}, \lambda)$$

is a category $\mathcal{C}$ with

- a tensor bifunctor and a dualising functor,
- isomorphisms α, σ, Φ, and ∂,
- a virtual unit functor $\mathbb{I} \colon \mathcal{C} \to \mathrm{SET}$, and
- a left virtual unit natural isomorphism

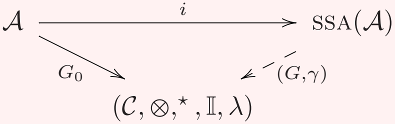
$$\lambda_{A,B} \colon \mathbb{I}(A^\star \otimes B) \cong \hom(A, B)$$

satisfying the associativity pentagon, the symmetry hexagon, and the four coherence axioms $\boldsymbol{\lambda\sigma}^\star$, $\boldsymbol{\lambda\alpha}^\star$, $\otimes\boldsymbol{\sigma}$, and $\otimes\boldsymbol{\alpha}$

Main theorem

Theorem

The category of proof nets over atomic formulae $\mathcal{A}$ is the free semi-$\star$-autonomous category $\mathrm{SSA}(\mathcal{A})$



Semi-$\star$-autonomous functors

A semi-$\star$-autonomous functor

$$(G, \gamma) \ : \ (\mathcal{C}, \otimes, ^{\star}, \mathbb{I}, \lambda_{\mathcal{C}}) \ \rightarrow \ (\mathcal{D}, \otimes, ^{\star}, \mathbb{J}, \lambda_{\mathcal{D}})$$

consists of

- a functor $G \colon \mathcal{C} \rightarrow \mathcal{D}$ preserving the tensor and duality functors
- a natural transformation $\gamma \colon \mathbb{I} \Rightarrow \mathbb{J}G$

satisfying the following, equivalent conditions:

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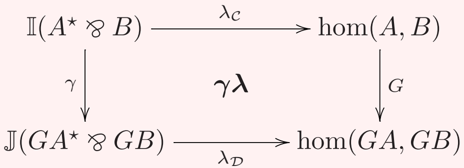
- a functor $G : \mathcal{C} \to \mathcal{D}$ preserving the tensor and duality functors
- a natural transformation $\gamma : \mathbb{I} \Rightarrow \mathbb{J}G$

satisfying the following, equivalent conditions:

(1)

$$G(\lambda_{\mathcal{C}}(x)) = \lambda_D(\gamma(x))$$



Semi-$\star$-autonomous functors

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consists of

- a functor $G\colon \mathcal{C} \to \mathcal{D}$ preserving the tensor and duality functors
- a natural transformation $\gamma\colon \mathbb{I} \Rightarrow \mathbb{J}G$

satisfying the following, equivalent conditions:

(2)

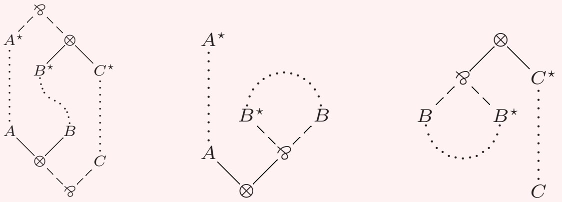
$$G(x \otimes g) = \gamma(x) \otimes Gg$$

$$\begin{array}{ccc}
\mathbb{I}X & \xrightarrow{\;-\otimes f\;} & \hom(A, X \otimes B) \\
{\scriptstyle\gamma}\downarrow & \boldsymbol{\gamma\otimes} & \downarrow{\scriptstyle G} \\
\mathbb{J}GX & \xrightarrow[\;-\otimes Gf\;]{} & \hom(GA, GX \otimes GB)
\end{array}$$

Related work

The approach via linearly distributive categories

A category with tensor, duality, associativity and symmetry, plus:



switch* $\qquad$ η' $\qquad$ ϵ'

* or *dissociativity*, or *weak* or *linear distributivity*

This approach is equivalent (both describe categories of proof nets)

See [Cockett & Seely 1991/1997] and [Došen & Petrić, 2005]

The approach via promonoidal categories

A promonoidal category has tensor and unit profunctors

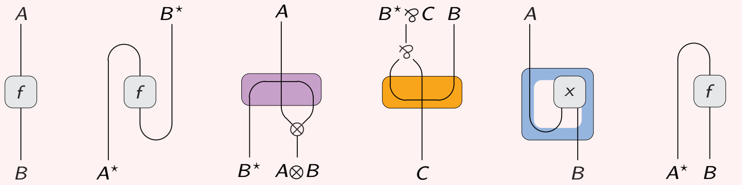
$$P : A \times A \nrightarrow A \qquad\qquad J : 1 \nrightarrow A$$

Idea: let P be represented by an actual tensor bifunctor, but not J
When fully carried out, this approach would be essentially the same as ours, since the profunctor J is a functor $J : \mathcal{C} \to \text{SET}$

See [Robin Houston's Ph.D. thesis, 2008], and several drafts and technical reports from 2005 by Robin Houston, Dominic Hughes, and Andrea Schalk

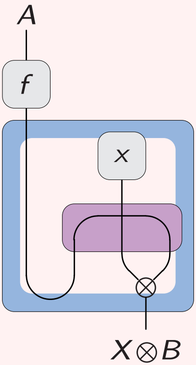
Wire diagrams

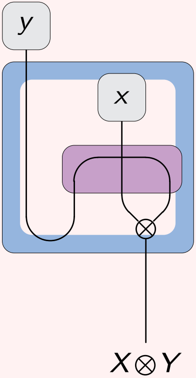
Wire diagram components



f $\quad f^\star$ $\quad \eta$ $\quad \epsilon$ $\quad \lambda(x)$ $\quad \lambda^{-1}(f)$

Virtual tensors in wire diagrams

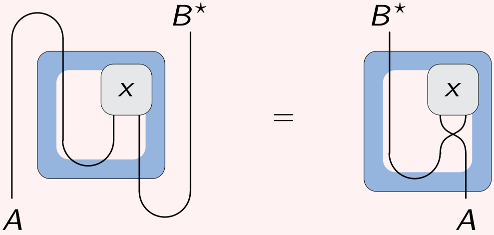




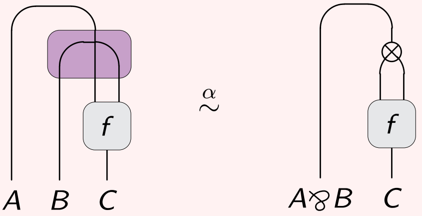
$$x \otimes f = \lambda(\mathbb{I}(\eta)(x)) \circ f$$

$$x \otimes y = \mathbb{I}(x \otimes id)(y)$$

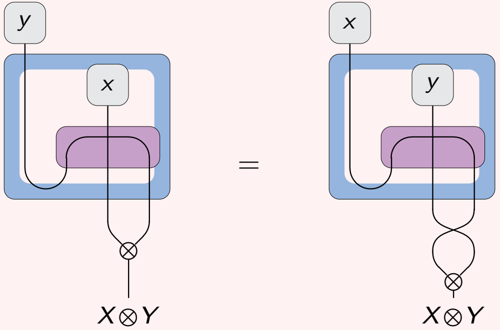
$$\begin{array}{ccc}
\mathbb{I}(A \parr B) & \xrightarrow{\mathbb{I}(\sigma^\star)} & \mathbb{I}(B \parr A) \\
\Big\downarrow \lambda & \boldsymbol{\lambda\sigma}^\star & \Big\downarrow \lambda \\
\hom(A^\star, B) & \xrightarrow[(-)^\star]{} & \hom(B^\star, A)
\end{array}$$

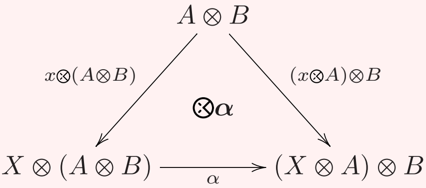


$$
\begin{array}{ccc}
\mathbb{I}((A \parr B) \parr C)) & \xrightarrow{\mathbb{I}(\alpha^\star)} & \mathbb{I}(A \parr (B \parr C) \\
\Big\downarrow{\scriptstyle\lambda} & \boldsymbol{\lambda\alpha}^\star & \Big\downarrow{\scriptstyle\lambda} \\
\hom(A^\star \otimes B^\star, C) & \xrightarrow[\Phi]{} & \hom(A^\star, B \parr C)
\end{array}
$$

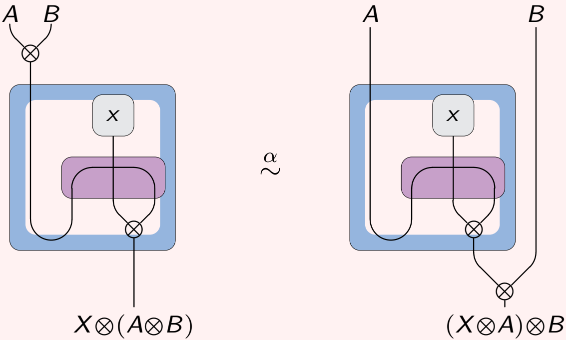


$$\begin{array}{ccc} \mathbb{I}A \times \mathbb{I}B & \xrightarrow{\sigma} & \mathbb{I}B \times \mathbb{I}A \\ {\scriptstyle -\otimes-}\big\downarrow & \boldsymbol{\otimes\sigma} & \big\downarrow{\scriptstyle -\otimes-} \\ \mathbb{I}(A \otimes B) & \xrightarrow[\mathbb{I}\sigma]{} & \mathbb{I}(B \otimes A) \end{array}$$





$A \otimes B$
$x \otimes (A \otimes B)$
$(x \otimes A) \otimes B$
$\otimes\alpha$
$X \otimes (A \otimes B)$
α
$(X \otimes A) \otimes B$



A
B
x
$X \otimes (A \otimes B)$
$\overset{\alpha}{\sim}$
A
B
x
$(X \otimes A) \otimes B$

Proving the main theorem

Tree-sequents

A tree-sequent is a binary tree with annotated formulae for leaves

$$t \;:=\; A_V \mid (t, t)$$

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From t, a sequent is extracted by $\lfloor t \rfloor$

$$\lfloor A_V \rfloor = \{A_V\} \qquad \lfloor (s, t) \rfloor = \lfloor s \rfloor \uplus \lfloor t \rfloor$$

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A $\mathcal{C}$-object is extracted by $\bigotimes t$

$$\bigotimes(A_V) = A$$

$$\bigotimes(s,t) = (\bigotimes s) \otimes (\bigotimes t)$$

Tree-sequents

A tree-sequent is a binary tree with annotated formulae for leaves
A tree-context with a hole allows manipulation inside tree-sequents

$$t \;:=\; A_V \mid (t, t) \qquad\qquad t\{x\} \;:=\; \{x\} \mid (t, t\{x\}) \mid (t\{x\}, t)$$

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A $\mathcal{C}$-object is extracted by $\bigotimes t$

$$\bigotimes(A_V) = A \qquad\qquad \bigotimes\{t\} = \bigotimes t$$

$$\bigotimes(s, t) = (\bigotimes s) \otimes (\bigotimes t) \qquad\qquad \bigotimes\{f\} = f$$

Coherence for the tensor

A coherence isomorphism from s to t is a map

$$f\colon \bigotimes s \to \bigotimes t \qquad\qquad (\ \lfloor s \rfloor = \lfloor t \rfloor\ )$$

constructed by composition, inversion, and identity from

$$\bigotimes t\{\sigma\}\ :\ \bigotimes t\{r, s\} \to \bigotimes t\{s, r\}$$

$$\bigotimes t\{\alpha\}\ :\ \bigotimes t\{q, (r, s)\} \to \bigotimes t\{(q, r), s\}$$

Coherence [MacLane]: for any s, t such that $\lfloor s \rfloor = \lfloor t \rfloor$ there is exactly one coherence isomorphism from s to t.

Two-sided tree-sequents

A two-sided tree-sequent $s? \triangleright t$ is of the form $\triangleright t$ or $s \triangleright t$, and has an associated sequent

$$\lfloor \triangleright t \rfloor = \lfloor t \rfloor \qquad \lfloor s \triangleright t \rfloor = \lfloor s \rfloor^{\star} \uplus \lfloor t \rfloor$$

and an associated real or virtual hom-object in SET

$$S(\triangleright t) = \mathbb{I}(\bigotimes t) \qquad S(s \triangleright t) = \mathrm{hom}(\bigotimes s, \bigotimes t)$$

Equivariance

An equivariance isomorphism from $q? \triangleright r$ to $s? \triangleright t$ is an isomorphism in SET

$$f \; : \; S(q? \triangleright r) \to S(s? \triangleright t)$$

built by composition (and identity) from:

- isomorphisms Φ, Φ^{-1}, λ, and λ^{-1}, and functor $-^{\star}$
- $(- \circ f)$, $(f^{\star} \circ -)$, and $\mathbb{I}(f^{\star})$ for coherence isomorphisms f

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- $(- \circ f)$, $(f^{\star} \circ -)$, and $\mathbb{I}(f^{\star})$ for coherence isomorphisms f

Equivariance: for any two tree-sequents with the same associated sequent there is exactly one equivariance isomorphism

A sequent calculus for SSA morphisms

$$\frac{}{a_V \triangleright a_W}\ id_a \qquad \frac{q? \triangleright r}{s? \triangleright t}\ \nu(-)$$

where ν is an equivariance iso from $(q? \triangleright r)$ to $(s? \triangleright t)$

$$\frac{s\{A_V, B_W\} \triangleright t}{s\{A_V \otimes_u B_W\} \triangleright t} = \qquad \frac{s? \triangleright t\{A_V, B_W\}}{s? \triangleright t\{A_V \parr_u B_W\}} =$$

$$\frac{s \triangleright A_V \qquad t \triangleright B_W}{(s, t) \triangleright A_V \otimes_u B_W}\ (-\otimes-) \qquad \frac{\triangleright A_V \qquad t \triangleright B_W}{t \triangleright A_V \otimes_u B_W}\ (-\otimes-)$$

$$\frac{s \triangleright A_V \qquad \triangleright B_W}{s \triangleright A_V \otimes_u B_W}\ (-\otimes-) \qquad \frac{\triangleright A_V \qquad \triangleright B_W}{\triangleright A_V \otimes_u B_W}\ (-\otimes-)$$

$$\frac{}{A_V \triangleright A_W}\ id_A \qquad \frac{s \triangleright A_V \qquad A_V \triangleright t}{s \triangleright t}\ (-\circ-) \qquad \frac{\triangleright A_V \qquad A_V \triangleright t}{\triangleright t}\ \mathbb{I}(-)(-)$$

The virtual tensor as a cut

$$\dfrac{\overline{\overline{\rhd X}}^{\,x} \qquad \overline{\overline{A \rhd B}}^{\,f}}{A \rhd X \otimes B}\,(-\otimes-) \qquad \sim \qquad \dfrac{\dfrac{\overline{\overline{\rhd X}}^{\,x} \qquad \dfrac{\dfrac{\overline{\overline{X \rhd X}}^{\,id} \qquad \overline{\overline{A \rhd B}}^{\,f}}{X, A \rhd X \otimes B}\,(-\otimes-)}{X \rhd A^\star, X \otimes B}\,\Phi}{\rhd A^\star, X \otimes B}\,\mathbb{I}(-)(-)}{A \rhd X \otimes B}\,\lambda$$

$$x \otimes f \qquad\qquad \lambda(\mathbb{I}(\Phi(id \otimes f))(x))$$

Proving the main theorem

A proof of $s? \triangleright t$ in the calculus on the previous slide constructs

- a map $f\colon \bigotimes s \to \bigotimes t$ or a virtual map $x \in \bigotimes t$
- a proof net $\mathcal{L} \blacktriangleright \lfloor s? \triangleright t \rfloor$

(by collecting axiom links of cut-free proofs, and applying composition on a cut)

Two proofs construct the same proof net if and only if they are equal up to cut-elimination and permutations

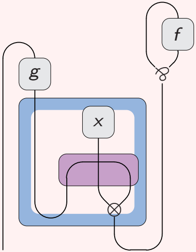
To show: two proofs construct the same map if and only if they are equal up to cut-elimination and permutations

A cut-elimination step

$$\dfrac{\dfrac{\overline{\overline{A \triangleright C^\star}}\;f}{\triangleright A^\star \parr C^\star}\;{-}^{-1} \qquad \dfrac{\dfrac{\overline{\overline{\triangleright A}}\;x \qquad \overline{\overline{B \triangleright C}}\;g}{B \triangleright A \otimes C}\;(-\otimes-)}{A^\star \parr C^\star \triangleright B^\star}\;(-)^\star}{\triangleright B^\star}\;\mathbb{I}(-)(-) \quad\rightsquigarrow\quad \dfrac{\dfrac{\overline{\overline{\triangleright A}}\;x \qquad \overline{\overline{A \triangleright C^\star}}\;f}{\triangleright C^\star}\;\mathbb{I}(-)(-) \qquad \dfrac{\overline{\overline{B \triangleright C}}\;g}{C^\star \triangleright B^\star}\;(-)^\star}{\triangleright B^\star}\;\mathbb{I}(-)(-)$$

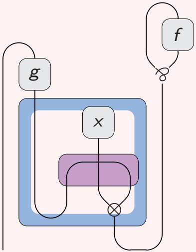
A cut-elimination step

$$\dfrac{\dfrac{\overline{\overline{A \triangleright C^\star}}\ f}{\triangleright A^\star \parr C^\star}\ {-}^{-1} \qquad \dfrac{\dfrac{\overline{\overline{\triangleright A}}\ x \qquad \overline{\overline{B \triangleright C}}\ g}{B \triangleright A \otimes C}\ (-\otimes-)}{A^\star \parr C^\star \triangleright B^\star}\ (-)^\star}{\triangleright B^\star}\ \mathbb{I}(-)(-) \quad \rightsquigarrow \quad \dfrac{\dfrac{\overline{\overline{\triangleright A}}\ x \qquad \overline{\overline{A \triangleright C^\star}}\ f}{\triangleright C^\star}\ \mathbb{I}(-)(-) \qquad \dfrac{\overline{\overline{B \triangleright C}}\ g}{C^\star \triangleright B^\star}\ (-)^\star}{\triangleright B^\star}\ \mathbb{I}(-)(-)$$



A cut-elimination step

$$\dfrac{\dfrac{\overline{\overline{}}\;f}{\dfrac{A \triangleright C^\star}{\triangleright A^\star \parr C^\star}}\,{-}^{-1} \qquad \dfrac{\dfrac{\dfrac{\overline{\overline{\triangleright A}}\;x \qquad \overline{\overline{B \triangleright C}}\;g}{B \triangleright A \otimes C}\,(-\otimes-)}{A^\star \parr C^\star \triangleright B^\star}\,(-)^\star}{}\;}{\triangleright B^\star}\,\mathbb{I}(-)(-) \quad \rightsquigarrow \quad \dfrac{\dfrac{\overline{\overline{\triangleright A}}\;x \qquad \overline{\overline{A \triangleright C^\star}}\;f}{\triangleright C^\star}\,\mathbb{I}(-)(-) \qquad \dfrac{\overline{\overline{B \triangleright C}}\;g}{C^\star \triangleright B^\star}\,(-)^\star}{\triangleright B^\star}\,\mathbb{I}(-)(-)$$



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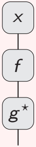
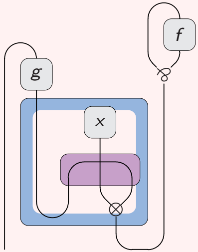
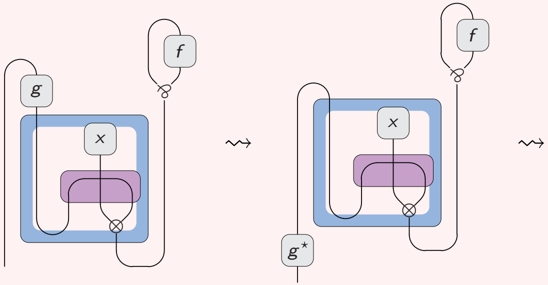
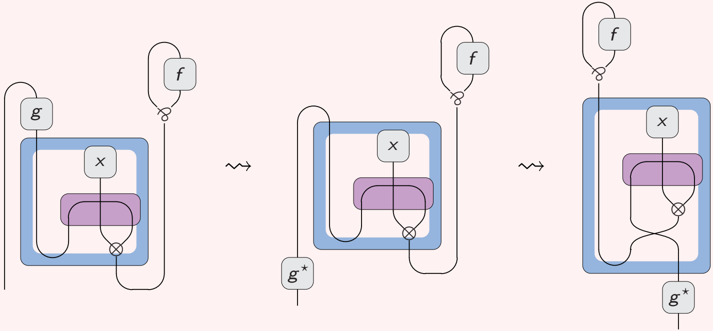




$\rightsquigarrow$

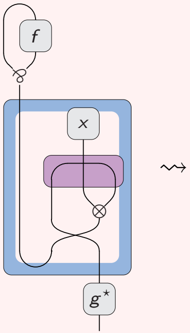


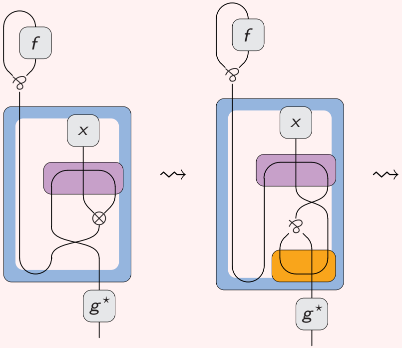
$$(h \circ g)^\star = g^\star \circ h^\star$$



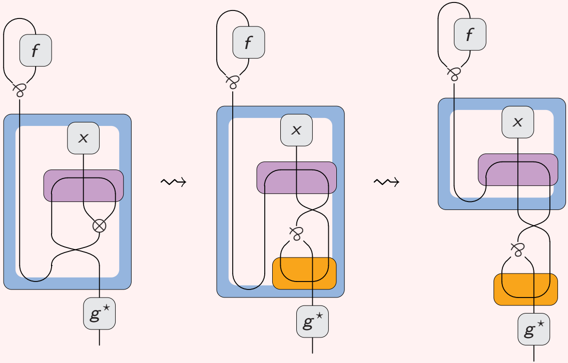
$$(h \circ g)^\star = g^\star \circ h^\star$$

$$\begin{array}{ccc} \mathbb{I}(A \parr B) & \xrightarrow{\mathbb{I}(\sigma^\star)} & \mathbb{I}(B \parr A) \\ \lambda \downarrow & \boldsymbol{\lambda\sigma^\star} & \downarrow \lambda \\ \hom(A^\star, B) & \xrightarrow[(-)^\star]{} & \hom(B^\star, A) \end{array}$$


f
x
g*

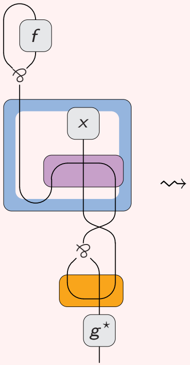


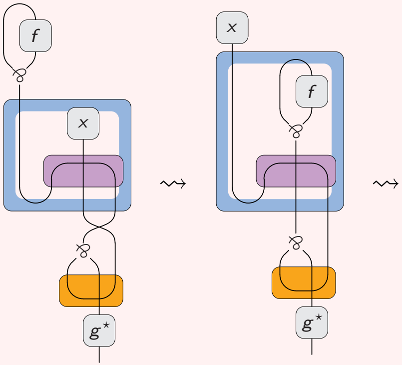
$$
\begin{array}{ccc}
\hom(A \otimes B, C^\star) & \xrightarrow{(-)^\star} & \hom(C, A^\star \parr B^\star) \\
\downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi^{-1}} \\
\hom(A, B^\star \parr C^\star) & \boldsymbol{\Phi\sigma} & \hom(C \otimes A, B^\star) \\
\downarrow{\scriptstyle \sigma^\star \circ -} & & \downarrow{\scriptstyle - \circ \sigma} \\
\hom(A, C^\star \parr B^\star) & \xleftarrow[\Phi]{} & \hom(A \otimes C, B^\star)
\end{array}
$$



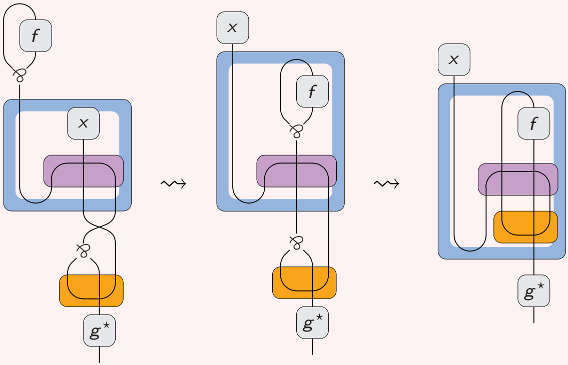
$$
\begin{array}{ccc}
\hom(A \otimes B, C^\star) & \xrightarrow{(-)^\star} & \hom(C, A^\star \parr B^\star) \\
\downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi^{-1}} \\
\hom(A, B^\star \parr C^\star) & \boldsymbol{\Phi\sigma} & \hom(C \otimes A, B^\star) \\
\downarrow{\scriptstyle \sigma^\star \circ -} & & \downarrow{\scriptstyle - \circ \sigma} \\
\hom(A, C^\star \parr B^\star) & \xleftarrow[\Phi]{} & \hom(A \otimes C, B^\star)
\end{array}
$$

naturality of λ


f
x
g⋆

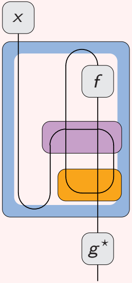


$$\begin{array}{ccc} \mathbb{I}A \times \mathbb{I}B & \xrightarrow{\sigma} & \mathbb{I}B \times \mathbb{I}A \\ {\scriptstyle -\otimes-}\big\downarrow & \otimes\boldsymbol{\sigma} & \big\downarrow{\scriptstyle -\otimes-} \\ \mathbb{I}(A \otimes B) & \xrightarrow[\mathbb{I}\sigma]{} & \mathbb{I}(B \otimes A) \end{array}$$

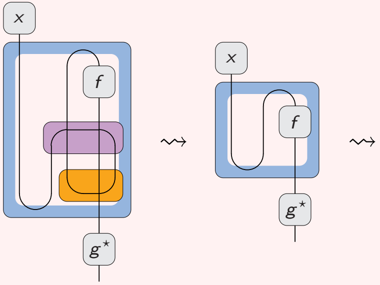


$$\begin{array}{ccc}
\mathbb{I}A \times \mathbb{I}B & \xrightarrow{\sigma} & \mathbb{I}B \times \mathbb{I}A \\
\Big\downarrow {\scriptstyle -\otimes-} & \otimes\sigma & \Big\downarrow {\scriptstyle -\otimes-} \\
\mathbb{I}(A \otimes B) & \xrightarrow[\mathbb{I}\sigma]{} & \mathbb{I}(B \otimes A)
\end{array}$$

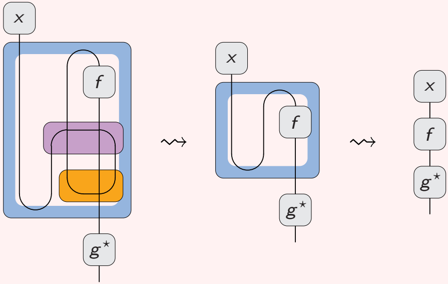
naturality of λ



$\rightsquigarrow$



$$(id \otimes \epsilon) \circ \eta = id$$



$$(id \parr \epsilon) \circ \eta = id \qquad \lambda \circ \lambda^{-1} = id$$

Conclusions

The virtual unit allows function application in categories

(other than by composition with a point $I \to A$)

Then semi-$\star$-autonomous categories are easily characterised

(such that proof nets describe the free one)