Linear Functors and their Fixed Points

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Introduction

- Linear actegories: A linearly distributive category with a monoidal category acting on it both covariantly and contravariantly.
  -The Logic of Message Passing (J. R. B. Cockett and Craig Pastro)
- We shall prove that the actions give the structure of a parameterized linear functor and the inductive and coinductive data types form a linear functor pair (when data is built on a linear functor).
- In particular, circuit diagrams are helpful to establish these facts.

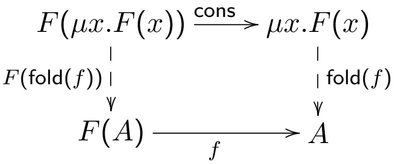
Motivation

- The logic of products and coproducts gives the logic of communication along channel.
- Linearly distributive categories manage communication channels.
- Linear actegories provide message passing in process world.
- Linear functor gives a basis on which one can build inductive (and coinductive) concurrent data or protocols.

Algebraic definition of Inductive datatype

An inductive datatype for an endo-functor $F : \mathbb{X} \to \mathbb{X}$ is:

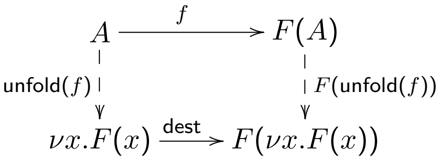
- An object $\mu x.F(x)$.
- A map cons : $F(\mu x.F(x)) \to \mu x.F(x)$ such that given any object $A \in \mathbb{X}$ and a map $f : F(A) \to A$, there exists a unique fold map such that the following diagram commutes.



Algebraic definition of Coinductive datatype

Dually a coinductive datatype for F is:

- An object $\nu x.F(x)$.
- A map dest : $\nu x.F(x) \rightarrow F(\nu x.F(x))$ such that given any object $A \in \mathbb{X}$ and a map $f : A \rightarrow F(A)$, there exists a unique unfold map such that the following diagram commutes.



Fixed points

Lambek's Lemma

If $F : \mathbb{X} \to \mathbb{X}$ is a functor for which $\mu x.F(x)$ exists then cons : $F(\mu x.F(x)) \to \mu x.F(x)$ is an isomorphism and (dually) if $\nu x.F(x)$ exists then dest : $\nu x.F(x) \to F(\nu x.F(x))$ is an isomorphism.

Circular combinator (alternative method)

A (circular) combinator over F is

$$\frac{A \xrightarrow{f} D}{F(A) \xrightarrow{\mathsf{c}[f]} D}\ \mathsf{c}[\_]$$

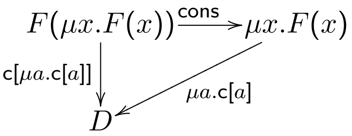
where

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & D & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} F(A) & \xrightarrow{F(h)} & F(A') \\ & {\scriptstyle \mathsf{c}[f]}\searrow \quad \swarrow{\scriptstyle \mathsf{c}[f']} & \\ & D & \end{array}$$

Circular definition of Inductive datatype

A circular inductive datatype is:

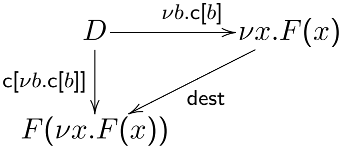
- An object $\mu x.F(x)$.
- A map $\mathsf{cons} : F(\mu x.F(x)) \to \mu x.F(x)$ such that given a (circular) combinator $\mathsf{c}[\_]$ over F, there exists a unique fold map $\mu a.\mathsf{c}[a]$ such that the following diagram commutes.



Circular definition of Coinductive datatype

Dually a circular coinductive datatype is:

- An object $\nu x.F(x)$.
- A map dest : $\nu x.F(x) \to F(\nu x.F(x))$ such that given a (circular) combinator c[_] over F, there exists a unique unfold map $\nu b.\mathsf{c}[b]$ such that the following diagram commutes.



Circular rules

We can express cons, dest, fold and unfold in proof theoretically.

- fold map

$$\dfrac{\begin{array}{|lr} \hline \forall X & f : X \to D \\ \hline & \\ \multicolumn{2}{|c}{\dfrac{X \to D}{F(X) \to D}} \\ \hline \end{array}}{\mu x.F(x) \to D}$$

- unfold map

$$\dfrac{\begin{array}{|lr|} \hline \forall X & f : D \to X \\ \hline & \\ \multicolumn{2}{|c|}{\dfrac{D \to X}{D \to F(X)}} \\ \hline \end{array}}{D \to \nu x.F(x)}$$

Circular rules

- cons

$$\frac{X \xrightarrow{f} F(\mu x.F(x))}{X \xrightarrow{\mathsf{cons}[f]} \mu x.F(x)}$$

- dest

$$\frac{F(\nu x.F(x)) \xrightarrow{f} X}{\nu x.F(x) \xrightarrow{\mathsf{dest}[f]} X}$$

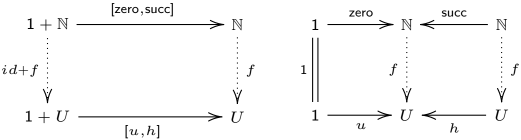
- These circular rules are used to form datatypes.

Example for inductive datatype

- The set of natural numbers $\mathbb{N}$ with zero and succ constructors

$$1 + \mathbb{N} \xrightarrow{[\text{zero},\text{succ}]} \mathbb{N}$$

- This map forms an inductive datatype for natural numbers such that the following diagram commutes.



- If we use circular combinator, then

$$\cfrac{\forall X \qquad X \vdash_f \mathbb{N}}{\cfrac{1 \vdash_{\text{zero}} \mathbb{N} \qquad X \vdash_{\text{succ}(X)} \mathbb{N}}{1 + X \vdash \mathbb{N}}}$$

$$\mathbb{N} \vdash_g \mathbb{N}$$

Polycategories

- A **Polycategory** $\mathbb{X}$ is a category that consists of list of objects with polymaps.
- For example, $P, Q, R \vdash A, B, C$.
- These maps correspond to Gentzen sequents.
- Composition of polymaps is the cut rules. For example,

$$\frac{P, Q \vdash R, A \qquad A, B \vdash C, D}{P, Q, B \vdash R, C, D}$$

Representability of $\otimes$ and $\oplus$

We can represent $\otimes$ and $\oplus$ by sequents calculus rules of inference. For example,

$$\frac{\Gamma_1, X, Y, \Gamma_2 \vdash \Delta}{\Gamma_1, X \otimes Y, \Gamma_2 \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta_1, X, Y, \Delta_2}{\Gamma \vdash \Delta_1, X \oplus Y, \Delta_2}$$

$$\frac{\Gamma_1, X \vdash \Delta_1 \quad Y, \Gamma_2 \vdash \Delta_2}{\Gamma_1, X \oplus Y, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

$$\frac{\Gamma_1 \vdash \Delta_1, X \quad \Gamma_2 \vdash Y, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, X \otimes Y, \Delta_2}$$

Linear distribution

- A representable polycategory gives us linearly distributive category.
- For example, a derivation of one linear distribution is

$$\dfrac{\dfrac{\dfrac{X \vdash X \quad Y \vdash Y}{X, Y \vdash X \otimes Y} \quad Z \vdash Z}{X, Y \oplus Z \vdash X \otimes Y, Z}}{X \otimes (Y \oplus Z) \vdash (X \otimes Y) \oplus Z}$$

Symmetric linearly distributive category

A linearly distributive category is symmetric if both the tensors and pars are symmetric. For symmetric case, there are two linear distributions.

$$
\begin{aligned}
\delta_R^L : A \otimes (B \oplus C) &\rightarrow B \oplus (A \otimes C) \\
\delta_L^R : (B \oplus C) \otimes A &\rightarrow (B \otimes A) \oplus C
\end{aligned}
$$

that must satisfy some coherence conditions. For example,

$$
\begin{aligned}
\delta_R^L;\ 1 \oplus a_\otimes &= a_\otimes;\ 1 \otimes \delta_R^L;\ \delta_R^L \\
\delta_L^R;\ \delta_R^L \oplus 1;\ a_\oplus &= \delta_R^L;\ 1 \oplus \delta_L^R
\end{aligned}
$$

Circular rules for linearly distributive categories

- Circular rules are natural formalism to get fixed points in linearly distributive categories.
- If we have closure, then

$$
\dfrac{\dfrac{\boxed{\begin{array}{l} \forall X \qquad\qquad X \vdash \Gamma \Rightarrow \Delta \\ \hline \dfrac{X \vdash \Gamma \Rightarrow \Delta}{F(X) \vdash \Gamma \Rightarrow \Delta}\ \mathsf{c}[\_] \end{array}}}{\mu x.F(x) \vdash \Gamma \Rightarrow \Delta}}{\Gamma, \mu x.F(x) \vdash \Delta}
$$

  But it is not expressable in the linearly distributive setting.
- Circular rules allow us to express this

$$
\dfrac{\boxed{\begin{array}{l} \forall X \qquad\qquad \Gamma, X \vdash \Delta \\ \hline \dfrac{\Gamma, X \vdash \Delta}{\Gamma, F(X) \vdash \Delta}\ \mathsf{c}[\_] \end{array}}}{\Gamma, \mu x.F(x) \vdash \Delta}
$$

Monoidal functor

- Suppose $F : \mathbb{X} \to \mathbb{X}$ is a monoidal functor.
- So there must be the following two natural transformations.
  - $m_\otimes : F(A) \otimes F(B) \to F(A \otimes B)$
  - $m_\top : \top \to F(\top)$

  that must satisfy two equations.
  - $(m_\top \otimes 1)\ m\ F(u) = u$
  - $a_\otimes\ (1 \otimes m)\ m = (m \otimes 1)\ m\ F(a_\otimes)$

Is the greatest fixed point of a monoidal functor monoidal?

Proposition

The greatest fixed point of a monoidal functor is monoidal and dually the least fixed point of a comonoidal functor is comonoidal.

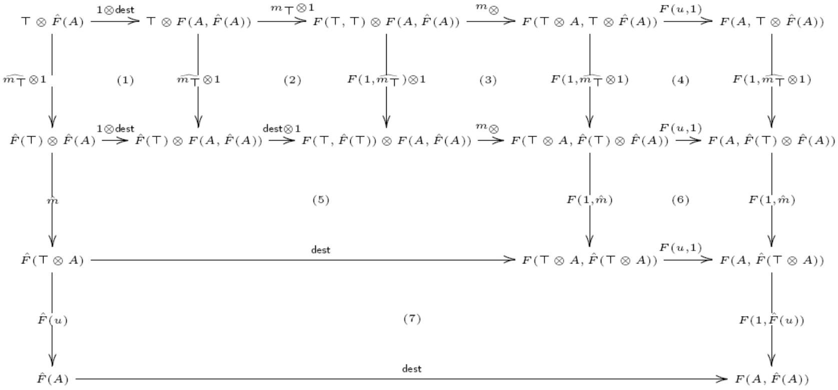
- Consider $\hat{F} = \nu x.F(\_, x)$ is the greatest fixed point of a monoidal functor.
- To prove that $\hat{F}$ is monoidal, we have to show that the two equations hold.
- Consider the first equation, $(\widehat{m_\top} \otimes 1)\ \widehat{m}\ \widehat{F}(u) = u$
- It suffices to show that for a fixed g, $(\widehat{m_\top} \otimes 1)\ \widehat{m}\ \widehat{F}(u) = \text{unfold}(g)$ and $u = \text{unfold}(g)$.

Defining diagram of $\widehat{m}$ and $\widehat{m_\top}$

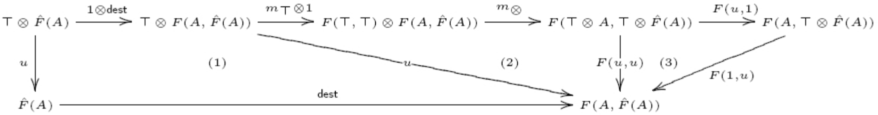
$$\begin{array}{ccccc}
\widehat{F}(A) \otimes \widehat{F}(B) & \xrightarrow{\mathsf{dest} \otimes \mathsf{dest}} & F(A, \widehat{F}(A)) \otimes F(B, \widehat{F}(B)) & \xrightarrow{m_\otimes} & F(A \otimes B, \widehat{F}(A) \otimes \widehat{F}(B)) \\
 & \searrow^{\widehat{m}} & & & \downarrow{\scriptstyle F(1, \widehat{m})} \\
 & & \widehat{F}(A \otimes B) & \xrightarrow[\mathsf{dest}]{} & F(A \otimes B, \widehat{F}(A \otimes B))
\end{array}$$

$$\begin{array}{ccc}
\top & \xrightarrow{m_\top} & F(\top, \top) \\
{\scriptstyle \widehat{m_\top}}\downarrow & & \downarrow{\scriptstyle F(1, \widehat{m_\top})} \\
\widehat{F}(\top) & \xrightarrow[\mathsf{dest}]{} & F(\top, \widehat{F}(\top))
\end{array}$$

$(\widehat{m_\top} \otimes 1)\ \hat{m}\ \widehat{F}(u) = \mathsf{unfold}[(m_\top \otimes \mathsf{dest})\ m_\otimes\ F(u,1)]$



$u = \mathsf{unfold}[(m_\top \otimes \mathsf{dest})\ m_\otimes\ F(u, 1)]$



- So $(\widehat{m_\top} \otimes 1)\ \widehat{m}\ \widehat{F}(u) = u$
- The greatest fixed point of a monoidal functor is monoidal.

Linear Functor

- A linear functor is a functor that consists of a monoidal $(F : \mathbb{X} \to \mathbb{Y})$ and a comonoidal $(\bar{F} : \mathbb{X} \to \mathbb{Y})$ functor and four natural transformations (called "linear strengths").

$$
\begin{aligned}
v_{\otimes}^{R} &: F(A \oplus B) \to \bar{F}(A) \oplus F(B) \\
v_{\otimes}^{L} &: F(A \oplus B) \to F(A) \oplus \bar{F}(B) \\
v_{\oplus}^{R} &: F(A) \otimes \bar{F}(B) \to \bar{F}(A \otimes B) \\
v_{\oplus}^{L} &: \bar{F}(A) \otimes F(B) \to \bar{F}(A \otimes B)
\end{aligned}
$$

- The above data must satisfy several coherence conditions. For example,

$$
\begin{aligned}
(m_{\otimes} \otimes 1)\ v_{\oplus}^{R}\ \bar{F}(a_{\otimes}) = a_{\otimes}\ (1 \otimes v_{\oplus}^{R})\ v_{\oplus}^{R} \\
(v_{\otimes}^{L} \otimes 1)\ \delta_{R}^{R}\ (1 \oplus v_{\oplus}^{L}) = m_{\otimes}\ F(\delta_{R}^{R})\ v_{\otimes}^{L} \\
(v_{\otimes}^{R} \otimes 1)\ \delta_{R}^{R}\ (1 \oplus m_{\otimes}) = m_{\otimes}\ F(\delta_{R}^{R})\ v_{\otimes}^{R}
\end{aligned}
$$

Linear fixed point

Proposition

The fixed point of a linear functor is linear.

In order to prove this, we have to show that

- The greatest fixed point of a monoidal functor, $\hat{F}$ is monoidal and (dually) the least fixed point of a comonoidal functor, $\bar{\hat{F}}$ is comonoidal.(Proved)
- There exist linear strengths between these two fixed point functors that must satisfy the coherence conditions.

Does linear strength exist?

- Prove $\hat{F}(A) \otimes \bar{\hat{F}}(B) \vdash_{\hat{v}^R_{\oplus}} \bar{\hat{F}}(A \otimes B)$ map exists and it is unique fold map.
- It suffices to show that if there is a combinator c[_]

$$\frac{\hat{F}(A) \otimes X \vdash \bar{\hat{F}}(A \otimes B)}{\hat{F}(A) \otimes \bar{F}(B, X) \vdash \bar{\hat{F}}(A \otimes B)} \text{ c[\_]}$$

$\hat{v}^R_\oplus$ map exists

$$\forall X \qquad \hat{F}(A)\otimes X \vdash_f \bar{\hat{F}}(A\otimes B)$$

$$\dfrac{A\otimes B \vdash_{id} A\otimes B \quad \hat{F}(A)\otimes X \vdash_f \bar{\hat{F}}(A\otimes B)}{\bar{F}(A\otimes B, \hat{F}(A)\otimes X) \vdash_{\bar{F}(1,f)} \bar{F}(A\otimes B, \bar{\hat{F}}(A\otimes B))}$$

$$\dfrac{\bar{F}(A\otimes B, \hat{F}(A)\otimes X) \vdash_{\bar{F}(1,f)} \bar{F}(A\otimes B, \bar{\hat{F}}(A\otimes B))}{\bar{F}(A\otimes B, \hat{F}(A)\otimes X) \vdash_{\bar{F}(1,f);\mathsf{cons}} \bar{\hat{F}}(A\otimes B)}$$

$$\dfrac{\bar{F}(A\otimes B, \hat{F}(A)\otimes X) \vdash_{\bar{F}(1,f);\mathsf{cons}} \bar{\hat{F}}(A\otimes B)}{F(A,\hat{F}(A))\otimes \bar{F}(B,X) \vdash_{v^R_\oplus;\bar{F}(1,f);\mathsf{cons}} \bar{\hat{F}}(A\otimes B)}$$

$$\dfrac{F(A,\hat{F}(A))\otimes \bar{F}(B,X) \vdash_{v^R_\oplus;\bar{F}(1,f);\mathsf{cons}} \bar{\hat{F}}(A\otimes B)}{\hat{F}(A)\otimes \bar{F}(B,X) \vdash_{\mathsf{dest}\otimes 1;v^R_\oplus;\bar{F}(1,f);\mathsf{cons}} \bar{\hat{F}}(A\otimes B)}$$

$$\dfrac{\forall X \quad \hat{F}(A)\otimes X \vdash_f \bar{\hat{F}}(A\otimes B) \quad \Longrightarrow \quad \hat{F}(A)\otimes \bar{F}(B,X) \vdash_{\mathsf{dest}\otimes 1;v^R_\oplus;\bar{F}(1,f);\mathsf{cons}} \bar{\hat{F}}(A\otimes B)}{\hat{F}(A)\otimes \bar{\hat{F}}(B) \vdash_{\hat{v}^R_\oplus} \bar{\hat{F}}(A\otimes B)}$$

- So there exists $\hat{v}^R_\oplus$.
- $\hat{v}^R_\oplus$ is unique fold map such that $1\otimes \mathsf{cons}; \hat{v}^R_\oplus = \mathsf{c}[\hat{v}^R_\oplus] = \mathsf{dest}\otimes 1; v^R_\oplus; \bar{F}(1,\hat{v}^R_\oplus); \mathsf{cons}$.

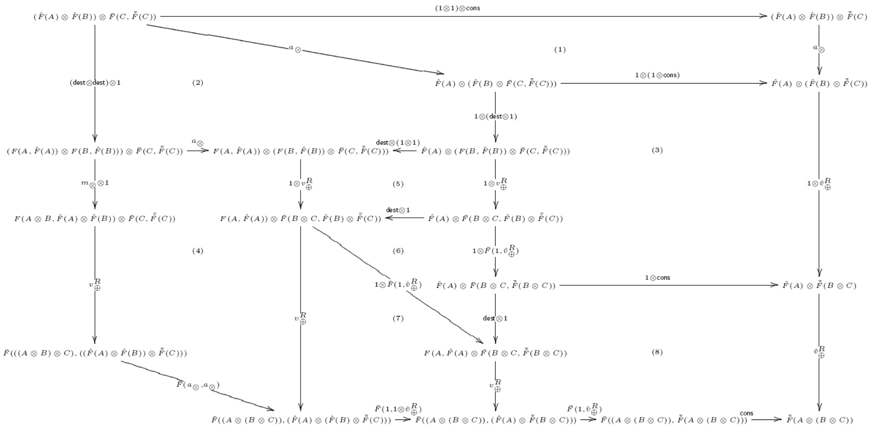
Coherence condition

- Linear strengths must satisfy the coherence conditions. For example,

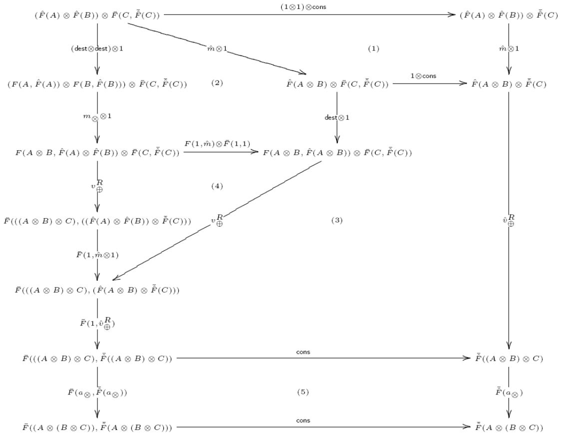
$$(\hat{m} \otimes 1)\ \hat{v}^R_\oplus\ \bar{\hat{F}}(a_\otimes) = a_\otimes\ (1 \otimes \hat{v}^R_\oplus)\ \hat{v}^R_\oplus$$

- It suffices to show that they both equal to fold map that means it suffices to find a combinator u[ ] such that
  - $((1 \otimes 1) \otimes \mathsf{cons})\ a_\otimes\ (1 \otimes \hat{v}^R_\oplus)\ \hat{v}^R_\oplus = \mathsf{u}[a_\otimes\ (1 \otimes \hat{v}^R_\oplus)\ \hat{v}^R_\oplus]$
  - $((1 \otimes 1) \otimes \mathsf{cons})\ (\hat{m} \otimes 1)\ \hat{v}^R_\oplus\ \bar{\hat{F}}(a_\otimes) = \mathsf{u}[(\hat{m} \otimes 1)\ \hat{v}^R_\oplus\ \bar{\hat{F}}(a_\otimes)]$

$((1 \otimes 1) \otimes \mathsf{cons})\ a_\otimes\ (1 \otimes \hat{v}^R_\oplus)\ \hat{v}^R_\oplus = \mathsf{u}[a_\otimes\ (1 \otimes \hat{v}^R_\oplus)\ \hat{v}^R_\oplus]$



$((1 \otimes 1) \otimes \mathsf{cons})\ (\hat{m} \otimes 1)\ \hat{v}^R_\oplus\ \bar{\hat{F}}(a_\otimes) = \mathsf{u}[(\hat{m} \otimes 1)\ \hat{v}^R_\oplus\ \bar{\hat{F}}(a_\otimes)]$



- $(\hat{m} \otimes 1)\ \hat{v}^R_\oplus\ \bar{\hat{F}}(a_\otimes) = a_\otimes\ (1 \otimes \hat{v}^R_\oplus)\ \hat{v}^R_\oplus$ holds.
- So if a linear functor has linear fixed point then it is linear.

Linear Actegories

- A linearly distributive category with a monoidal category acting on it both covariantly and contravariantly is called linear actegories.
- Linear $\mathbb{A}$- actegory is:

$$\circ : \mathbb{A} \times \mathbb{X} \to \mathbb{X} \quad \text{and} \quad \bullet : \mathbb{A}^{op} \times \mathbb{X} \to \mathbb{X}.$$

- Here $\mathbb{A} = (\mathbb{A}, *, I, a_*, l_*, r_*, c_*)$ is a symmetric monoidal category and $\mathbb{X}$ is a symmetric linear distributive category.
- The two "actions" of $\mathbb{A}$ on $\mathbb{X}$ are $\circ$ and $\bullet$.
- The unit and counit are denoted by $n_{A,X} : X \to A \bullet (A \circ X)$ and $e_{A,X} : A \circ (A \bullet X) \to X$.

Linear Actegories

- The natural isomorphisms in $\mathbb{X}$ for all $A, B \in \mathbb{A}$ and $X, Y \in \mathbb{X}$

$$u_\circ : I \circ X \to X,$$
$$u_\bullet : X \to I \bullet X,$$
$$a_\circ^* : (A * B) \circ X \to A \circ (B \circ X),$$
$$a_\bullet^* : A \bullet (B \bullet X) \to (A * B) \bullet X,$$
$$a_\otimes^\circ : A \circ (X \otimes Y) \to (A \circ X) \otimes Y,$$
$$a_\oplus^\bullet : (A \bullet X) \oplus Y \to A \bullet (X \oplus Y).$$

- The natural morphisms in $\mathbb{X}$ for all $A, B \in \mathbb{A}$ and $X, Y \in \mathbb{X}$

$$d_\oplus^\circ : A \circ (X \oplus Y) \to (A \circ X) \oplus Y,$$
$$d_\otimes^\bullet : (A \bullet X) \otimes Y \to A \bullet (X \otimes Y),$$
$$d_\bullet^\circ : A \circ (B \bullet X) \to B \bullet (A \circ X)$$

Linear Actegories

- The above data must satisfy some coherence conditions. For example,

$$
\begin{gathered}
a^*_\circ \; (A \circ d^\circ_\bullet) \; d^\circ_\bullet = d^\circ_\bullet \; (C \bullet a^*_\circ) \\
d^\circ_\bullet \; (A \bullet d^\circ_\bullet) \; a^*_\bullet = (C \circ a^*_\bullet) \; d^\circ_\bullet \\
a^\circ_\otimes \; (a^\circ_\otimes \otimes Z) \; a_\otimes = (A \circ a_\otimes) \; (a^\circ_\otimes) \\
a_\oplus \; (a^\bullet_\oplus \oplus Z) \; a^\bullet_\oplus = a^\bullet_\oplus \; (A \bullet a_\oplus)
\end{gathered}
$$

Actions $\Rightarrow$ Linear functor?

Proposition

$A \bullet \_$ and $A \circ \_$ give the structure of a linear functor.

In order to prove this, we have to show that

- $A \bullet \_$ is a monoidal functor and $A \circ \_$ is a comonoidal functor.
- “Linear strengths“ exist that must satisfy the coherence conditions.

Is $A \bullet \_$ a monoidal functor?

- For a functor to be monoidal, there are two natural transformations

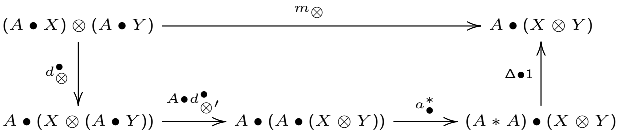
$$m_\otimes : (A \bullet X) \otimes (A \bullet Y) \to A \bullet (X \otimes Y)$$
$$m_\top : \top \to (A \bullet \top)$$

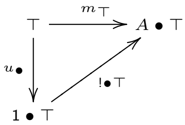
- These must satisfy two equations.

$$\begin{aligned} l_\otimes &= (m_\top \otimes 1)\; m_\otimes\; (A \bullet l_\otimes) \\ a_\otimes\; (1 \otimes m_\otimes)\; m_\otimes &= (m_\otimes \otimes 1)\; m_\otimes\; (A \bullet a_\otimes) \end{aligned}$$

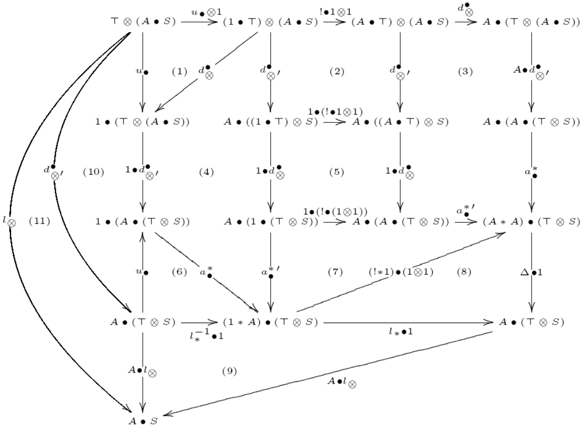
- To prove $A \bullet \_$ is a monoidal functor, we have to show that the above two equations hold.

Defining diagram of $m_\otimes$ and $m_\top$





$l_\otimes = (m_\top \otimes 1)\; m_\otimes\; (A \bullet l_\otimes)$

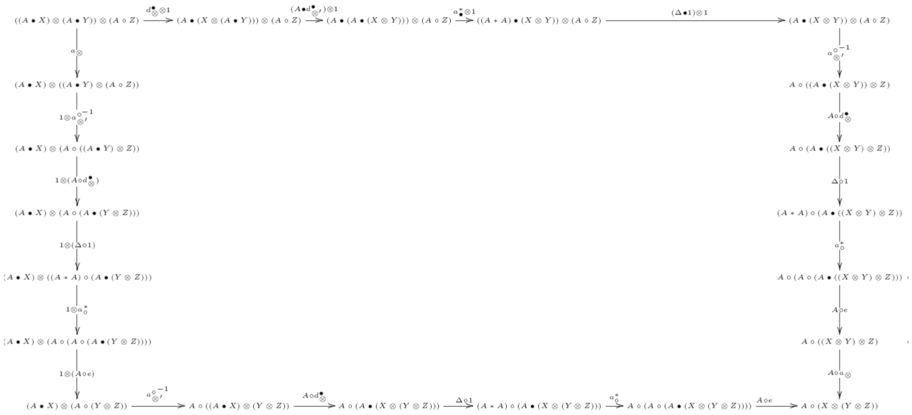


- So $A \bullet \_$ is a monoidal functor and dually $A \circ \_$ is a comonoidal functor.

Linear strengths

- Consider one linear strength $v^R_\oplus : (A \bullet X) \otimes (A \circ Y) \to A \circ (X \otimes Y)$ that must satisfy the coherence conditions.
- For example, $(m_\otimes \otimes 1)\ v^R_\oplus\ (A \circ a_\otimes) = a_\otimes\ (1 \otimes v^R_\oplus)\ v^R_\oplus$
- $v^R_\oplus = a^{\circ^{-1}}_{\otimes'}; A \circ d^\bullet_\otimes; \Delta \circ 1; a^*_\circ; A \circ e$
- $m_\otimes = d^\bullet_\otimes; A \bullet d^\bullet_{\otimes'}; a^*_\bullet; \Delta \bullet 1$

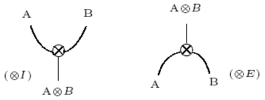
$(m_\otimes \otimes 1)\, v^R_\oplus \,(A \circ a_\otimes) = a_\otimes\, (1 \otimes v^R_\oplus)\, v^R_\oplus$



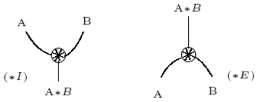
- Difficult to show categorically...
- Circuit diagrams are easier and they do have to satisfy the net conditions.

Circuit Rules

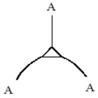
- Circuit introduction and elimination rules for $\otimes$



- Circuit introduction and elimination rules for $*$

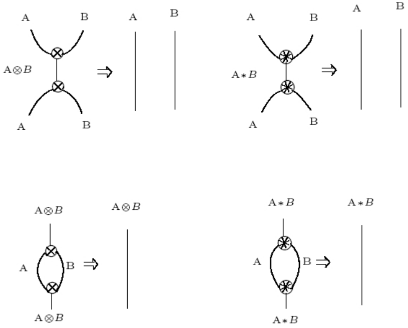


- Copy rule



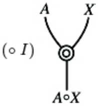
Circuit Rules

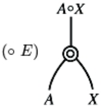
- Circuit reduction rules for $\otimes$ and $*$



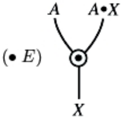
Circuit Rules

- Circuit introduction and elimination rules for $\circ$



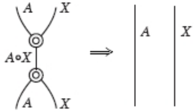


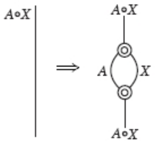
- Circuit elimination rule for $\bullet$



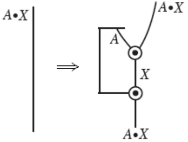
Circuit Rules

- Circuit reduction and expansion rules for $\circ$



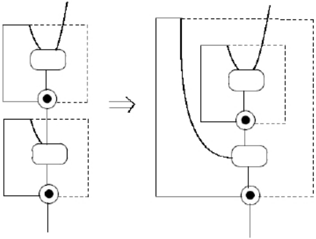


- Circuit expansion rule for $\bullet$

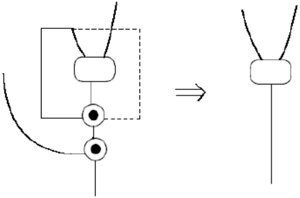


Circuit Rules

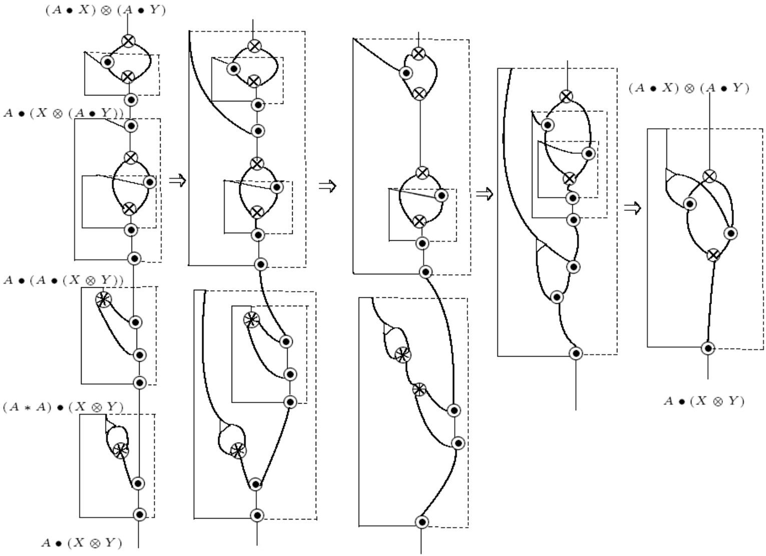
- Box-eats-box rule



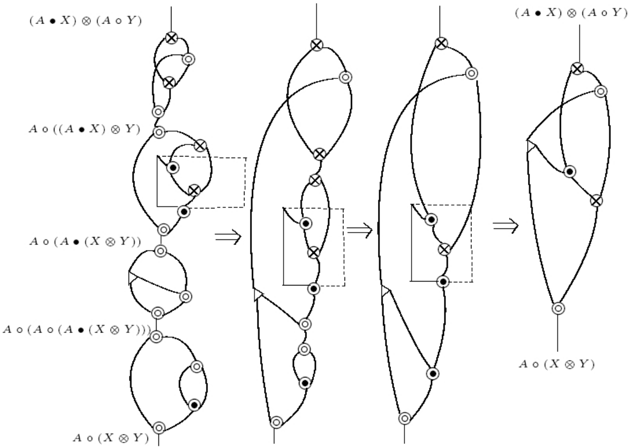
- Box-elimination rule



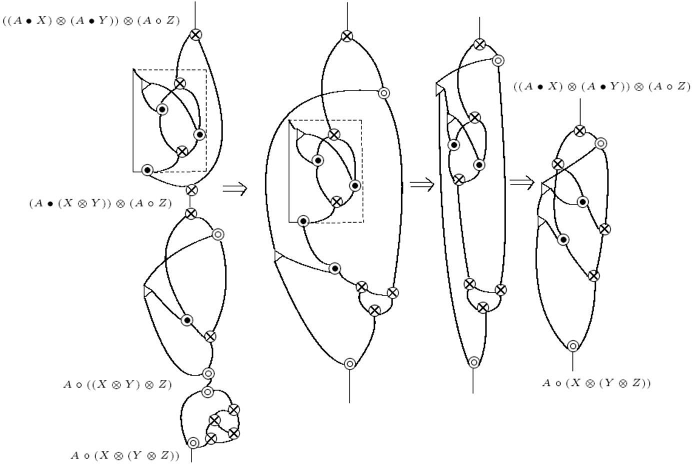
Circuit Diagram of $m_{\otimes}$ for $\bullet$



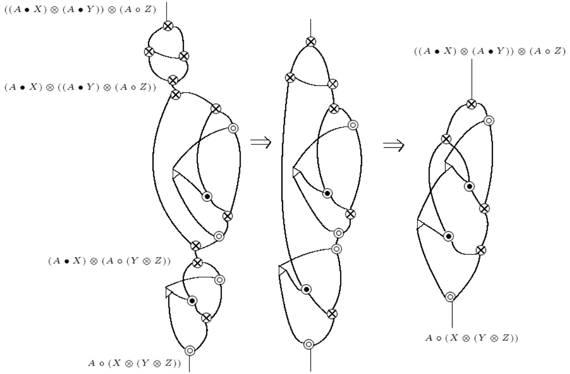
Circuit Diagram for $v^R_{\oplus}$



Circuit Diagram for $[(m_{\otimes} \otimes 1)\ v_{\oplus}^{R}\ (A \circ a_{\otimes})]$



Circuit Diagram for $[a_\otimes\ (1 \otimes v_\oplus^R)\ v_\oplus^R]$



- So $(m_\otimes \otimes 1)\ v_\oplus^R\ (A \circ a_\otimes) = a_\otimes\ (1 \otimes v_\oplus^R)\ v_\oplus^R$.
- $A \bullet \_$ and $A \circ \_$ give the structure of a linear functor.

Conclusion

- The greatest fixed point of a monoidal functor is monoidal.
- The fixed point of a linear functor is linear.
- The actions of linear actegories give the structure of a parameterized linear functor.

Thank you