# Operations, Effects and Monads for the $\pi$-Calculus

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# I. Lawvere theories

and

Notions of Computation

# From Lawvere Theories to Side-Effects [Power, Plotkin 2000 – ]

Universal algebras $\rightsquigarrow$ Monads $\rightsquigarrow$ Effects

Lawvere theory: Category $L$ with arities for objects and operations for arrows $op : p \to q$ $(\mathbb{N}^{op} \subset L)$

A model is a product-preserving $M : L \to C$

Equational theory: Collection of operations $\Sigma$ with arities $op : A^p \to A^q$ and equations $E$ between them.

A model in category $C$ is an object $A$ with arrows + diagrams

Monads: $U : Mod(L, Set) \to Set$ has a left adjoint with induced monad $T$ such that $Mod(L, Set) \simeq T\text{-}Alg$.

For any set $X$ this gives $op_X : (TX)^p \to (TX)^q$ algebraic.

Effects: Algebraic operations correspond to constants in the Kleisli category; generic effects $eff_{op} : q \to Tp$

Plotkin + Power extend to countable, enriched arities ($\omega Cpo$) and give a range of examples in computational monads.

## Example: Global Store (i)

Given sets of locations $L$ and values $V$, effects for store are $\mathrm{lookup} : L \to TV$ and $\mathrm{update} : L \times V \to T1$

The appropriate monad is $TX = (S \times X)^S$ where $S = V^L$ with lookup and update defined in an obvious way.

This generates algebraic operations

$$l_X : (TX)^V \to (TX)^L \quad \text{and} \quad u_X : TX \to (TX)^{L \times V}$$

given by

$$l(M)_v = (\text{let } v = !l \text{ in } M)_l$$

$$u(M) = (l := v; M)_{l,v}.$$

# Example: Global store (ii)

[P,P; FoSSaCS '02]

In a category $C$ with finite products, a GS-algebra is an object $A$ and maps $u : A \to A^{L \times V}$ $\ell : A^V \to A^L$ satisfying seven equations:

$$\ell_l(u_{l,v}(x))_v = x$$

etc.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "A^!"'] & A^{L\times V} \arrow[r, "\sim"] & (A^V)^L \arrow[d, "\ell^L"] \\
A^L & A^{L\times L} \arrow[l, "A^\Delta"'] & (A^L)^L \arrow[l, "\sim"']
\end{tikzcd}$$

There is a forgetful $\mathcal{U} : (A, u, \ell) \mapsto A$ into $C$, exhibiting:

<u>Theorem</u> The category $GS(C)$ of GS-algebras is monadic over $C$, with monad $T(-) = (S \times -)^S$ for $S = V^L$.

Moreover, the corresponding effects are the familiar

$$\text{lookup} : L \to TV \quad \text{and} \quad \text{update} : L \times V \to T1 .$$

## Other Examples

Nondeterminism:

Operation - choice : $A^2 \longrightarrow A$ + assoc., comm., idem.

Monad - $\mathbb{P}_{fin+}(-)$ nonempty finite powerset

Generic effect - $t+f : 1 \to T2 = \mathbb{P}(2)$

Input / Output

Operations - read : $A^I \longrightarrow A$, write : $A \longrightarrow A^O$

Monad - $\mu Y.((-) + O \times Y + Y^I)$

Generic effects - $r : 1 \to T I \quad w : O \to T1$

Exceptions

Operations - throw : $1 \to A^E$

Monad - $(-) + E$

Generic effect - $th : E \to T0$

# Operations and Computational Effects

Specifying a signature of operations and equations upon them gives a way to describe notions of computation

## Good news:

- We can from this derive computational monads <u>and</u> generic effects. [Plotkin, Power 2001]
- Combining theories is simple and flexible. [Hyland, Plotkin, Power 2002]
- Enriched arities gives $\omega$Cpo models [Plotkin, Power 2003]

## Other news:

- Only for monads of countable rank (so not $R^{R^A}$).
- Only for constructor operations, not destructors like exception handling (yet).

II. $\pi$ - calculus,

$Set^{II}$

# The $\pi$-calculus

We take finite $\pi$-calculus processes, given by

$$P ::= \bar{x}y.P \mid x(y).P \mid \tau.P$$
$$\mid 0 \mid P+Q \mid P|Q$$
$$\mid \nu x P \mid [x=y]P \mid [x\neq y]P$$

Transition semantics $P \xrightarrow{\alpha} P$ and bisimilarity $P \sim Q$ are defined as usual.

The plan is to use Plotkin + Power's enriched Lawvere theories to present an equational theory of $\pi$. The monad and free model that arises is that of [Fiore, Moggi, Sangiorgi 96, S.96], fully abstract for bisimilarity.

## Category $\mathbf{Set}^{\mathbb{I}}$

We work in the functor category $\mathbf{Set}^{\mathbb{I}}$, where $\mathbb{I}$ is finite sets and injections. This has both cartesian and monoidal closed structure:

$$A \times B\,(k) = A(k) \times B(k)$$

$$B^A(k) = [A(k+-), B(k+-)]$$

$$A \otimes B\,(k) = \int^{k'+k'' \hookrightarrow k} A(k') \times B(k'')$$

$$A \multimap B\,(k) = [A(-), B(k+-)]$$

$$A \otimes B \hookrightarrow A \times B \qquad (A \to B) \twoheadrightarrow (A \multimap B)$$

We take $\mathbf{Set}^{\mathbb{I}}$ as the category of arities*.

* or, a full skeletal subcategory of the countably presentable objects in $\mathbf{Set}^{\mathbb{I}}$

# More on $\mathbf{Set}^{\mathbb{I}}$

Object of names $\quad N(k) = k$

Endofunctor $\quad \delta A = A(\_+1)$

with, as it happens, $\delta A = N \multimap A$.

We also use the following maps, all natural in $A$ and $B$:

$A \xrightarrow{f} \delta A$ arising from $X \otimes Y \to Y$

$A^N \xrightarrow{g} \delta A$ -"- $X \otimes Y \to X \times Y$

$\delta(A^B) \xrightarrow{h} (\delta A)^B$ -"- $X \otimes (Y \times Z) \to (X \otimes Y) \times Z$

... where in fact $X$ is always $N$

With this structure, we also use $\mathbf{Set}^{\mathbb{I}}$ as the category for models and monads.

# III. Theory of $\pi$.

# Operations and effects for $\pi$

**I/O**

$\mathrm{out} : A \to A^{N \times N}$    output prefix $\bar{x}y.-$

$\mathrm{in} : A^N \to A^N$    input prefix $x(y).-$

$\mathrm{tau} : A \to A$    silent prefix $\tau.-$

**Nondeterminism**

$\mathrm{nil} : 1 \to A$    inactive process $0$

$\mathrm{choice} : A^2 \to A$    process sum $P + Q$

**Dynamic name creation**

$\mathrm{new} : \delta A \to A$    restriction $\nu x(-)$

**Corresponding computational effects:**

$\mathrm{send} : N \times N \to T1$    $\mathrm{die} : 1 \to T0$

$\mathrm{recv} : N \to TN$    $\mathrm{alt} : 1 \to T2$

$\mathrm{skip} : 1 \to T1$    $\mathrm{gensym} : 1 \to TN$

**Other operations:**

par is not algebraic (loosely, as $(P \mid Q);R \neq P;R \mid Q;R$)

$\mathrm{eq}, \mathrm{neq} : A \to A^{N \times N}$ are definable from $N \times N \cong N \otimes N + N$

$\mathrm{bout} : \delta A \to A^N$ can be defined from new, out.

# Component Equations

**I/O:** None

**Nondeterminism:** choice is associative, commutative and idempotent, with identity nil.

**Dynamic name creation:**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "id."'] & \delta A \arrow[d, "new"] \\ & A \end{tikzcd}$$

$$new \langle P \rangle_x = P$$

$$new \langle new \langle P \rangle_x \rangle_y = new \langle new \langle P \rangle_y \rangle_x$$

$$\begin{tikzcd} \delta^2 A \arrow[loop left, "twist"] \arrow[r, "\delta new"] & \delta A \arrow[r, "new"] & A \end{tikzcd}$$

# Combining Equations

Commuting:

$$\mathrm{new}\langle \mathrm{choice}(P,Q)\rangle_x = \mathrm{choice}(\mathrm{new}\langle P\rangle_x, \mathrm{new}\langle Q\rangle_y)$$

$$\begin{tikzcd}
\delta(A^2) \arrow[r, "\delta\mathrm{choice}"] \arrow[d, "g"'] & \delta A \arrow[r, "\mathrm{new}"] & A \arrow[d, "\mathrm{id}"] \\
(\delta A)^2 \arrow[r, "\mathrm{new}^2"] & A^2 \arrow[r, "\mathrm{choice}"] & A
\end{tikzcd}$$

$$\mathrm{new}\langle \mathrm{out}_{x,y}(P)\rangle_z = \mathrm{out}_{x,y}(\mathrm{new}\langle P\rangle_z) \qquad z \notin \{x,y\}$$

$$\mathrm{new}\langle \mathrm{in}_x(P)_y\rangle_z = \mathrm{in}_x(\mathrm{new}\langle P\rangle_z)_y \qquad z \notin \{x,y\}$$

$$\mathrm{new}\langle \mathrm{tau}(P)\rangle_z = \mathrm{tau}(\mathrm{new}\langle P\rangle_z)$$

Interaction

$$\mathrm{new}\langle \mathrm{out}_{x,y}(P)\rangle_x = \mathrm{nil}$$

$$\mathrm{new}\langle \mathrm{in}_x(P)_y\rangle_x = \mathrm{nil}$$

These are the equations that turn 'new' into restriction

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Category $PI(\mathrm{Set}^{I})$ of $\pi$-calculus models has objects $\langle A \in \mathrm{Set}^{I};\ \mathrm{out}, \mathrm{in}, \mathrm{tau}, \mathrm{choice}, \mathrm{nil}, \mathrm{new}\rangle$ + eqns with forgetful functor $U: PI(\mathrm{Set}^{I}) \to \mathrm{Set}^{I}$.

# Monads

Each component theory has a standard monad:

I/O $\qquad \mu Y. (X + N \times N \times Y + N \times Y^N + Y)$

Nondet. $\qquad \mathbb{P}_{fin}(X)$

Name generation $\quad \mathcal{D}yn(X) = \int^k X(\_ + k)$

Weaving these together as monad transformers gives

$$\mu Y. \mathbb{P}_{fin}(\mathcal{D}yn(X + N \times N \times Y + N \times Y^N + Y))$$

But this does not satisfy the interaction equations between new and in/out. The correct monad for $\pi$-algebras is

$$Pi(X) = \mu Y . \mathbb{P}_{fin}(\mathcal{D}yn(X) + N \times N \times Y + N \times \delta Y + N \times Y^N + Y)$$

which adds bound output but otherwise ignores name creation.

# Properties of the monad $Pi(X)$

1. Forgetful $\mathcal{U} : PI(Set^{\mathbb{I}}) \to Set^{\mathbb{I}}$ has a left adjoint $X \mapsto \langle Pi(X); \ldots \rangle$ making the category of $\pi$-algebras monadic over $Set^{\mathbb{I}}$.

2. $Pi(0)$ is the known fully abstract model of the (finite) $\pi$-calculus in $Set^{\mathbb{I}}$. In particular, it admits a standard definition of par by expansion.

3. As par is not algebraic, to define it on other $Pi(X)$ requires additional data specifying synchronisation of $X$ with input, output, tau and itself.

## Modalities

Each Lawvere theory gives rise to a modal logic over its algebras, with possibility and necessity modalities for each operation. For $\pi$, we get:

$$P \models \Diamond out_{x,y}(\phi)$$

$$\iff \exists Q.\ P \sim \bar{x}y.Q \wedge Q \models \phi$$

$$P \models \Box out_{x,y}(\phi)$$

$$\iff \forall Q.\ P \sim \bar{x}y.Q \Rightarrow Q \models \phi$$

$$P \models \Diamond choice(\phi, \psi)$$

$$\iff \exists Q, R.\ P \sim Q + R \wedge Q \models \phi \wedge R \models \psi$$

HML is definable

$$\langle \bar{x}y \rangle \phi = \Diamond choice(\Diamond out_{x,y}(\phi), true)$$

Alternatively, we could choose different operations to induce modalities closer to HML, like $(\bar{x}y.(-) + (=))$

In no case is there a $\phi \mid \psi$ modality.

## Summary

A Lawvere theory generated by operations and equations induces algebras and a monad.

Standard computational monads are instances of this, and each operation $A^p \to A^q$ has a matching effect $q \to Tp$.

Taking arities in $Set^{\mathbb{I}}$, we can give an equational theory for the $\pi$-calculus.

$$\boxed{\pi = \left(\text{I/O} + \text{choice} + \text{name creation}\right) \Big/ \text{new} \bowtie \text{I/O}}$$

The free algebra over $0$ is fully abstract for bisimilarity in the finite $\pi$-calculus.

The monad is almost, but not quite, the combination of its three components.

## What next?

- Use ωCpo for the full π-calculus
- Use partial order arities to constrain choice to lower or upper powerdomains. Link to Hennessy's testing models for π.
- Need proper theory of arities over two closed structures