Solving problems in topological groups (and number theory) using category theory

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Motivation

- (Bíró & Deshouillers & Sós, 2001) If H is a countable subgroup of $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, then $H = \{x \in \mathbb{T} \mid \lim n_k x = 0\}$ for some $\{n_k\} \subseteq \mathbb{Z}$.

Let $A \in \mathsf{Ab}(\mathsf{Haus})$.

- $\hat{A} := \mathscr{H}(A, \mathbb{T})$ (cts homomorphisms)

Dikranjan & Milan & Tonolo, 2005:

- $s_{\underline{u}}(A) := \{x \in A \mid \lim u_k(x) = 0 \text{ in } \mathbb{T}\}$ for $\underline{u} = \{u_n\} \subseteq \hat{A}$.
- $\mathfrak{g}_A(H) := \bigcap\{s_{\underline{u}}(A) \mid \underline{u} \in \hat{A}^{\mathbb{N}}, H \leq s_{\underline{u}}(A)\}$, where $H \leq A$.

If K is a compact Hausdorff abelian group, and $H \leq K$ is a countable subgroup, is $\mathfrak{g}_K(H) = H$?

Closure operators on $\mathsf{Grp}(\mathsf{Top})$

$\mathcal{G}$ is a full subcategory of $\mathsf{Grp}(\mathsf{Top})$, closed under subgroups.

- We use the $(\mathsf{Onto}, \mathsf{Embed})$ factorization system.
- $\operatorname{sub} G$ is the set of subgroups of $G \in \mathcal{G}$.

A **closure operator** c on $\mathcal{G}$ is a family of maps $(c_G \colon \operatorname{sub} G \to \operatorname{sub} G)_{G \in \mathcal{G}}$ such that:

- $S \subseteq c_G(S)$ for every $S \in \operatorname{sub} G$;
- $c_G(S_1) \subseteq c_G(S_2)$ whenever $S_1 \subseteq S_2$ and $S_i \in \operatorname{sub} G$;
- $f(c_{G_1}(S)) \subseteq c_{G_2}(f(S))$ whenever $f \colon G_1 \to G_2$ is a morphism in $\mathcal{G}$ and $S \in \operatorname{sub} G$.

Regular closure and groundedness

- c is **grounded** if $c_G(\{e\}) = \{e\}$ for every $G \in \mathcal{G}$.

Suppose that $\mathcal{G} \subseteq \mathsf{Ab}(\mathsf{Top})$.

- $\mathrm{reg}_G^{\mathcal{G}}(S) := \bigcap\{\ker f \mid S \subseteq \ker f, f\colon G \to G' \in \mathcal{G}\}$.
- c is grounded $\iff c_G(S) \leq \mathrm{reg}_G^{\mathcal{G}}(S)$ for every $S \in \operatorname{sub} G$ and $G \in \mathcal{G}$.

Examples:

- $\mathrm{reg}_G^{\mathsf{Ab}(\mathsf{Top})}(S) = S$.
- $\mathrm{reg}_G^{\mathsf{Ab}(\mathsf{Haus})}(S) = \mathrm{cl}_G\, S$.

Precompact abelian groups

- P is **precompact** if for every nbhd U of 0 there is a finite $F \subseteq P$ such that $F + U = P$. (Need not be Hausdorff!)

Comfort-Ross duality (1964): Let $A \in \mathsf{Ab}$ and $K := \hom(A, \mathbb{T})$.

- Monotone one-to-one correspondence between subgroups of K and precompact group topologies on A.
- $(H \leq K) \longmapsto$ initial topology with respect to $\Delta\colon A \to \mathbb{T}^H$.
- $(A, \tau) \longmapsto H = \widehat{(A, \tau)}$.

Precompact groups are pairs $P = (A, H)$, where $A = P_d$.
$f\colon (A_1, H_1) \to (A_2, H_2)$ is continuous $\iff \widehat{f}(H_2) \subseteq H_1$,
where $\widehat{f}\colon \hat{A}_2 \to \hat{A}_1$ is the dual of f.

Examples of precompact groups as pairs

- $\mathbb{T} = (\mathbb{T}_d, \mathbb{Z})$;
- $(\mathbb{Z}(p^\infty), \mathbb{Z})$, where $\mathbb{Z}(p^\infty)$ is a Prüfer group;
- $(\mathbb{Z}, \mathbb{Z}(p^\infty))$ is the integers with the p-adic topology;
- $(\mathbb{Z}, \langle\sqrt{2} + \mathbb{Z}\rangle)$ is the subgroup of $\mathbb{T}$ generated by $\sqrt{2}$;
- $(\mathbb{Z}, \mathbb{T}_d)$ is the Bohr-topology on $\mathbb{Z}$, that is, the finest precompact group topology on $\mathbb{Z}$.

CLOPs on AbHPr and functors on AbPr

- AbPr $=$ precompact abelian groups (with cts homo.).
- AbHPr $=$ precompact Hausdorff abelian groups.
- If $(A, H) \in$ AbPr, then
  - $\hat{A} \in$ AbHPr,
  - $H \in \operatorname{sub} \hat{A}$.
- Every closure operator c on AbHPr induces a functor
  - $C_c \colon$ AbPr $\longrightarrow$ AbPr
  - $C_c(A, H) = (A, c_{\hat{A}}(H))$ is a functor.
- C_c is a bicoreflection if and only if c is idempotent, that is, $c_G(H) = c_G(c_G(H))$.

The $\mathfrak{g}$ closure

- $f\colon X \to Y$ is **sequentially cts** if $x_n \longrightarrow x_0$ implies $f(x_n) \longrightarrow f(x_0)$.
- $P \in$ AbPr is an sk**-group** if every sequentially cts homomorphism $f\colon P \to K$ into a compact group is cts.

K a compact Hausdorff abelian group, $A = \hat{K}$ (discrete).

- $s_{\underline{u}}(K) := \{x \in K \mid \lim u_k(x) = 0 \text{ in } \mathbb{T}\}$ for $\underline{u} = \{u_n\} \subseteq A$.
- $\mathfrak{g}_K(H) := \bigcap\{s_{\underline{u}}(K) \mid \underline{u} \in A^{\mathbb{N}}, H \leq s_{\underline{u}}(K)\}$, where $H \leq K$.
- $\mathfrak{g}_K(H) = \{\chi\colon A \to \mathbb{T} \mid \chi$ sequentially cts on $(A, H)\}$.
- The bicoreflection $C_{\mathfrak{g}}$ maps (A, H) to the coarsest sk-group topology on A finer than H.

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Solution to the “motivational" problem:

- $\mathfrak{g}_K(H) = H \iff (A, H)$ is an sk-group.
- If H is countable, then $\mathbb{T}^H$ is metrizable, and (A, H) is a sequential space.

kk-groups

- $f\colon X \to Y$ is k**-cts** if $f_{|C}$ is cts for every compact C.
- $P \in \mathsf{AbPr}$ is a kk**-group** if every k-cts homomorphism $f\colon P \to K$ into a compact group is cts.

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- $\mathfrak{k}_K(H) := \{\chi\colon A \to \mathbb{T} \mid \chi\ k\text{-cts on } (A,H)\}$.

$P \in \mathsf{AbHPr}$, $K :=$ completion of P, and $H \leq P$.

- $\mathfrak{k}_P(H) := \mathfrak{k}_K(H) \cap P$.
- The bicoreflection $C_{\mathfrak{k}}$ maps (A,H) to the coarsest kk-group topology on A finer than H.
- Internal characterization of $\mathfrak{k} =$??

The G_δ-closure

- $f\colon X \to Y$ is **countably cts** if $f_{|C}$ is cts for every $|C| \le \omega$.
- G_δ-set $=$ a countable intersection of open sets.
- G_δ-topology $=$ topology whose base is the G_δ-sets.

For $P \in$ AbHPr and $S \le P$

- $\mathfrak{l}_P(S) =$ the closure of S in the G_δ-topology of P.

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- $\mathfrak{l}_K(H) = \{\chi\colon A \to \mathbb{T} \mid \chi$ countably cts on $(A, H)\}$.
- The following are equivalent:
  - $\mathfrak{l}_K(H) = H$;
  - H is realcompact;
  - every countably cts homomorphism from (A, H) into a compact group is continuous.

The G_δ-closure

K a compact Hausdorff abelian group, $A = \hat{K}$ (discrete).

- $\wr_K(H) = \{\chi \colon A \to \mathbb{T} \mid \chi$ countably cts on $(A, H)\}$.
- The following are equivalent:
  - $\wr_K(H) = H$;
  - H is realcompact;
  - every countably cts homomorphism from (A, H) into a compact group is continuous.
- If H is dense in K, the following are equivalent:
  - $\wr_K(H) = K$;
  - H is pseudocompact (cf. Comfort & Ross, 1966);
  - every homomorphism from A into a compact group is countably cts on (A, H).

Preservation of quotients (coequalizers)

Let c be a closure operator on AbHPr.
$P = (A, H) \in$ AbPr, $K := \hat{A}$, and $B \leq A$.

- $B^{\perp} := \{\chi \in K \mid \chi(B) = 0\}$, closed subgroup of K.
- $P/B = (A/B, H \cap B^{\perp})$ and $\widehat{A/B} \cong B^{\perp}$.
- $C_c(P/B) = C_c(P)/B \iff \boxed{c_{B^{\perp}}(H \cap B^{\perp}) = c_K(H) \cap B^{\perp}}$.
- $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{l}$ satisfy this condition.