# A biset-enriched categorical model for Proto-Quipper with dynamic lifting

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Quipper and Proto-Quipper are a family of quantum programming languages that, by their nature as circuit description languages, involve two runtimes: one at which the program generates a circuit and one at which the circuit is executed, normally with probabilistic results due to measurements. Accordingly, the language distinguishes two kinds of data: parameters, which are known at circuit generation time, and states, which are known at circuit execution time. Sometimes, it is desirable for the results of measurements to control the generation of the next part of the circuit. Therefore, the language needs to turn states, such as measurement outcomes, into parameters, an operation we call *dynamic lifting*. The goal of this paper is to model this interaction between the runtimes by providing a general categorical structure enriched in what we call "bisets". We demonstrate that the biset-enriched structure achieves a proper semantics of the two runtimes and their interaction, by showing that it models a variant of Proto-Quipper with dynamic lifting. The present paper deals with the concrete categorical semantics of this language, whereas a companion paper [7] deals with the syntax, type system, operational semantics, and abstract categorical semantics.

## 1 Introduction

Quipper [9, 10] is a functional programming language for designing quantum circuits. It shares many properties with hardware description languages. For example, Quipper distinguishes two kinds of runtime: (i) Circuit generation time. This is when a quantum circuit is generated on a classical computer. (ii) Circuit execution time. This is when a quantum circuit is run on a quantum computer or simulator. As a result of these two runtimes, Quipper makes a distinction between (i) *parameters* and (ii) *states*. A parameter is a value known at circuit generation time, such as a boolean for an if-then-else expression. A state is a value only known at circuit execution time, such as the state of a qubit or a bit in a circuit.

The distinction between parameters and states reflects the assumption that classical computers and quantum devices may reside in different physical locations and that they cooperate to perform computations. This is also an assumption shared by the quantum computing model QRAM [12]. In practice, the computation in a quantum device can interleave with the computation in a classical computer. This means that there should be a mechanism to turn the results of measurements, which are states, into parameters. Dynamic lifting is a construct that makes this possible in the programming language. It lifts the result of a measurement from a quantum computer to a boolean in the programming language, where it can then be used as a parameter in the construction of the rest of the circuit. This enables more general post-processing for quantum computation than the simpler model where all measurements are done at the end. Some quantum algorithms, such as those involving magic state distillation, require dynamic lifting, while many others do not.

Since Quipper is implemented as an embedded language in the host language Haskell, it does not have a formal semantics. Proto-Quipper [6, 8, 16, 17] is a family of quantum programming languages that are intended to provide Quipper with a formal foundation such as operational and categorical semantics. Like Quipper, Proto-Quipper has the two runtimes and distinguishes between parameters and states.

The semantics of the two runtimes depends on the meaning of "circuit" and "quantum operation". Rather than fixing one specific kind of circuit or quantum operation, the programming language is parametric on two categories $\mathbf{M}$ and $\mathbf{Q}$, which are assumed to be given but otherwise arbitrary, subject to some conditions. The first of these is a symmetric monoidal category $\mathbf{M}$, whose morphisms represent quantum circuits. The second is a symmetric monoidal category $\mathbf{Q}$, whose morphisms represent quantum operations. We note that there is an important conceptual difference between these categories. The morphisms of $\mathbf{M}$ represent circuits as *syntactic* entities. For example, Quipper allows circuits to be *boxed*, which turns them into a data structure that can be inspected and operated on. A boxed circuit may then be reversed, printed, iterated over, etc. Thus, $\mathbf{M}$ is typically a free category generated by some collection of (quantum and classical) gates. Measurement can be supported in the category $\mathbf{M}$, but it will merely be a gate in a circuit, turning a qubit into a classical bit of the circuit. On the other hand, the category $\mathbf{Q}$ represents quantum operations, which are *physical* entities. Typically, $\mathbf{Q}$ is a category of superoperators (which include unitary operations and measurements). We assume that $\mathbf{M}$ and $\mathbf{Q}$ have the same objects, and that there is an *interpretation functor* $J : \mathbf{M} \to \mathbf{Q}$, which interprets circuits by the quantum operations they embody.

We emphasize that measurement and dynamic lifting are two different concepts that should not be confused. Measurement is merely a gate in a quantum circuit, which turns a qubit (a state) into a classical bit (also a state). On the other hand, dynamic lifting is an operation of the programming language, which turns a classical bit (a state) into a boolean of the programming language (a parameter). In the categorical semantics, measurement is a morphism $\mathbf{Qubit} \to \mathbf{Bit}$ in the categories $\mathbf{M}$ and $\mathbf{Q}$. On the other hand, dynamic lifting is not a morphism in $\mathbf{M}$ or in $\mathbf{Q}$; rather, it is a morphism in a certain Kleisli category.

Specifically, in our recent work [7], we proposed a type system, an operational semantics and an abstract categorical semantics for a version of Proto-Quipper with dynamic lifting, which is called Proto-Quipper-Dyn. Dynamic lifting is modeled as a map $\mathbf{Bit} \to T\mathbf{Bool}$, where $T$ is a commutative strong monad, such that the following diagram commutes.

$$\begin{tikzcd} & \mathbf{Bit} \arrow[d, "\mathsf{dynlift}"] \\ \mathbf{Bool} \arrow[ur, "\mathsf{init}"] \arrow[r, "\eta"'] & T\mathbf{Bool} \end{tikzcd}$$

We have shown in [7] that our categorical model is sound with respect to the type system and operational semantics of the language. However, the categorical semantics in [7] is purely abstract, simply listing the properties that such a categorical model must have, without showing that such a category actually exists or giving an example of one.

In this paper, we construct a concrete model for the general categorical semantics of [7]. Constructing such a model is challenging because it requires a novel combination of quantum circuits (morphisms in $\mathbf{M}$) and quantum operations (morphisms in $\mathbf{Q}$): The categorical model must be able to account for both quantum circuits and quantum operations, as well as operations such as boxing, dynamic lifting, and of course higher-order functions.

Our technical innovation to make all of this work is *biset enrichment*. A *biset* is an object in the category $\mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$, or, more concretely, it is a triple $(X_0, X_1, f)$ of sets $X_0, X_1$ and a function $f : X_1 \to X_0$. A morphism of bisets is an obvious commutative square. Our construction is based on a biset-enriched category $\mathbf{C}$ constructed from $\mathbf{M}$ and $\mathbf{Q}$. Its objects are the same as those of $\mathbf{M}$ and $\mathbf{Q}$, and its hom-bisets are $(\mathbf{Q}(A,B), \mathbf{M}(A,B), J_{A,B})$, where the function $J_{A,B} : \mathbf{M}(A,B) \to \mathbf{Q}(A,B)$ is given by the interpretation functor $J$. A global element $f$ of $\mathbf{C}(A,B)$ consists of a pair of functions $f_0, f_1$ that makes the following diagram commute.

$$\begin{tikzcd} 1 \arrow[r, "f_1"] \arrow[dr, "f_0"'] & \mathbf{M}(A,B) \arrow[d, "J_{A,B}"] \\ & \mathbf{Q}(A,B) \end{tikzcd}$$

Thus, $f_1$ is a quantum circuit, which can be used as a quantum operation $f_0$ by composing with $J_{A,B}$. The biset-enriched category $\mathbf{C}$ therefore maintains a distinction between $\mathbf{M}$ and $\mathbf{Q}$ while taking the interpretation functor $J$ into account. To model the higher-order features of the programming language, we embed $\mathbf{C}$ in a monoidal closed biset-enriched category $\widetilde{\mathbf{C}}$, which we construct as a certain subcategory of the biset-enriched category of presheaves over $\mathbf{C}$. We show that $\widetilde{\mathbf{C}}$ satisfies the axiomatization specified in [7]. Therefore it is a concrete model for Proto-Quipper with dynamic lifting.

Our approach to modeling dynamic lifting differs from recent work by Lee et al. [14], where the category of *quantum channels*, which generalize quantum circuits with a notion of branching for measurement results, is used to model a single runtime. Because our model accounts separately for circuit generation time (category $\mathbf{M}$) and circuit execution time (category $\mathbf{Q}$), we are able to support a type system that distinguishes quantum circuits from quantum computations that use dynamic lifting [7]. This prevents a class of runtime errors in Quipper caused by boxing a computation that uses dynamic lifting.

The rest of the paper is structured as follows. In Section 2, we first review some basic concepts from enriched category theory, and then recall from [7] the axiomatization of an enriched categorical semantics for dynamic lifting. In Section 3, we define the biset-enriched category $\mathbf{C}$. We show its presheaf category $\overline{\mathbf{C}}$ admits a commutative strong monad and a linear-non-linear adjunction. In Section 4, we construct a reflective subcategory $\widetilde{\mathbf{C}}$ of $\overline{\mathbf{C}}$ and show that it is an enriched categorical model for dynamic lifting.

## 2 An enriched categorical semantics for dynamic lifting

Enriched categories are a generalization of categories where, instead of hom-sets, one works with hom-objects, which are objects in a monoidal category.

**Definition 2.1.** Let $\mathscr{V}$ be a monoidal category. A $\mathscr{V}$-enriched category $\mathbf{A}$ is given by the following:

- A class of objects, also denoted $\mathbf{A}$.
- For any $A,B \in \mathbf{A}$, an object $\mathbf{A}(A,B)$ in $\mathscr{V}$.
- For any $A \in \mathbf{A}$, a morphism $u_A : I \to \mathbf{A}(A,A)$ in $\mathscr{V}$, called the *identity* on $A$.
- For any $A,B,C \in \mathbf{A}$, a morphism $c_{A,B,C} : \mathbf{A}(A,B) \otimes \mathbf{A}(B,C) \to \mathbf{A}(A,C)$ in $\mathscr{V}$, called *composition*.
- The composition and identity morphisms must satisfy suitable diagrams in $\mathscr{V}$ (see [2, 11]).

**Remarks**

- Many concepts from the theory of non-enriched categories can be generalized to the enriched setting. For example, $\mathscr{V}$-functors, $\mathscr{V}$-natural transformations, $\mathscr{V}$-adjunctions, and the $\mathscr{V}$-Yoneda embedding are all straightforward generalizations of their non-enriched counterparts. We refer the reader to [2, 11] for comprehensive introductions. Symmetric monoidal categories can also be generalized to the enriched setting (see Appendix A for a definition).
- In the rest of this paper, when we speak of a map $f : A \to B$ in a $\mathscr{V}$-enriched category $\mathbf{A}$, we mean a morphism of the form $f : I \to \mathbf{A}(A,B)$ in $\mathscr{V}$. Furthermore, when $g : B \to C$ is also a map in $\mathbf{A}$, we write $g \circ f : A \to C$ as a shorthand for

$$I \xrightarrow{f \otimes g} \mathbf{A}(A,B) \otimes \mathbf{A}(B,C) \xrightarrow{c} \mathbf{A}(A,C).$$

- A $\mathscr{V}$-enriched category $\mathbf{A}$ gives rise to an ordinary (non-enriched) category $V(\mathbf{A})$, called the *underlying category* of $\mathbf{A}$.[1] The objects of $V(\mathbf{A})$ are the objects of $\mathbf{A}$ and the hom-sets of $V(\mathbf{A})$ are defined as $V(\mathbf{A})(A,B) = \mathscr{V}(I,\mathbf{A}(A,B))$, for any $A,B \in V(\mathbf{A})$. Similarly, a $\mathscr{V}$-functor $F : \mathbf{A} \to \mathbf{B}$ gives rise to a functor $VF : V(\mathbf{A}) \to V(\mathbf{B})$ and a $\mathscr{V}$-natural transformation $\alpha : F \to G$ gives rise to a natural transformation $V\alpha : VF \to VG$.

The construction in this paper is parameterized by two symmetric monoidal categories, denoted by $\mathbf{M}$ and $\mathbf{Q}$. We fix $\mathbf{M}$ and $\mathbf{Q}$ once and for all and require the following:

(1) $\mathbf{M}$ and $\mathbf{Q}$ have the same objects, including a distinguished object called $\mathbf{Bit}$. The category $\mathbf{M}$ has distinguished morphisms $\mathrm{zero}, \mathrm{one} : I \to \mathbf{Bit}$.

(2) $\mathbf{Q}$ has a coproduct $\mathbf{Bit} = I + I$, and the tensor product in $\mathbf{Q}$ distributes over this coproduct.

(3) There is a strict monoidal functor $J : \mathbf{M} \to \mathbf{Q}$ that is the identity on objects and $J(\mathrm{zero}) = \mathrm{inj}_1 : I \to I+I, J(\mathrm{one}) = \mathrm{inj}_2 : I \to I+I$. We call $J$ the *interpretation functor*.

(4) The category $\mathbf{Q}$ is enriched in *convex spaces*. That is, for any real numbers $p_1, p_2 \in [0,1]$ such that $p_1 + p_2 = 1$, and any maps $f,g \in \mathbf{Q}(A,B)$, there is a *convex sum* $p_1 f + p_2 g \in \mathbf{Q}(A,B)$, and the convex sum satisfies certain standard conditions which are detailed in Appendix C. Moreover, composition is *bilinear* with respect to convex sum, i.e., $(p_1 f_1 + p_2 f_2) \circ g = p_1(f_1 \circ g) + p_2(f_2 \circ g)$ and $h \circ (p_1 f_1 + p_2 f_2) = p_1(h \circ f_1) + p_2(h \circ f_2)$.

(5) For any $A \in \mathbf{Q}$, and $f : I \to \mathbf{Bit} \otimes A \in \mathbf{Q}$, we have $f = p_1(\mathrm{inj}_1 \otimes f_1) + p_2(\mathrm{inj}_2 \otimes f_2)$, where $\mathrm{inj}_1, \mathrm{inj}_2 : I \to I+I$ and $p_1, p_2 \in [0,1]$ are uniquely determined real numbers such that $p_1 + p_2 = 1$. When $p_i \neq 0$, the map $f_i : I \to A$ is also unique.

Perhaps it is useful to explain more specifically what we mean when we say that $\mathbf{M}$ and $\mathbf{Q}$ are fixed "once and for all". The point is that these categories are not only used in the categorical semantics, but also in the operational semantics of Proto-Quipper-Dyn (i.e., to run the program, we must know what a circuit is and what a quantum operation is). Therefore, these categories should be regarded as given as part of the language specification, rather than as a degree of freedom in the semantics. On the other hand, nothing in the operational or denotational semantics depends on particular properties of $\mathbf{M}$ and $\mathbf{Q}$ other than properties (1)–(5) above. Therefore, Proto-Quipper-Dyn can handle a wide variety of possible circuit models and physical execution models.

In practice, the category $\mathbf{M}$ will be a category of quantum circuits and the category $\mathbf{Q}$ will be a category of quantum operations. These categories will typically have additional objects, such as $\mathbf{Qubit}$ and perhaps $\mathbf{Qutrit}$, and additional morphisms, such as $H : \mathbf{Qubit} \to \mathbf{Qubit}$ and $\mathrm{Meas} : \mathbf{Qubit} \to \mathbf{Bit}$. Requirement (5) is only needed in the operational semantics of Proto-Quipper-Dyn; it is not needed for the denotational semantics.

We now recall the enriched categorical semantics for dynamic lifting specified in [7].

**Definition 2.2.** Let $\mathscr{V}$ be a cartesian closed category with coproducts. A $\mathscr{V}$-category $\mathbf{A}$ is a *model for Proto-Quipper with dynamic lifting* if it satisfies the following properties.

**a** $\mathbf{A}$ is symmetric monoidal closed, i.e., it is symmetric monoidal and there is a $\mathscr{V}$-adjunction $- \otimes A \dashv A \multimap -$ for any $A \in \mathbf{A}$.

**b** $\mathbf{A}$ has coproducts. Note that the tensor products distribute over coproducts, because $- \otimes A$ is a left adjoint functor for any $A \in \mathbf{A}$, which preserves coproducts.

[1] We use $V(\mathbf{A})$ to denote the underlying category, rather than the usual $U(\mathbf{A})$, because the letter $U$ will serve another purpose in this paper.

**c** **A** is equipped with a $\mathscr{V}$-adjunction $p : \mathscr{V} \to \mathbf{A} \dashv \flat : \mathbf{A} \to \mathscr{V}$ such that $p$ is a strong monoidal $\mathscr{V}$-functor. This implies that $p(1) \cong I$ and $p(X \times Y) \cong pX \otimes pY$.

**d** **A** is equipped with a commutative strong $\mathscr{V}$-monad $T$. For any $A, B \in \mathbf{A}$, we write $t_{A,B} : A \otimes TB \to T(A \otimes B)$ for the strength and $s_{A,B} : TA \otimes B \to T(A \otimes B)$ for the costrength.

**e** Let $V(\mathbf{A})$ be the underlying category of **A**, $VT$ be the underlying monad of $T$, and $Kl_{VT}(V(\mathbf{A}))$ be the Kleisli category of $VT$. The Kleisli category $Kl_{VT}(V(\mathbf{A}))$ is enriched in convex spaces. In other words, for any $A, B, C \in \mathbf{A}$, if $f, g : A \to TB$ and $p, q \in [0,1], p + q = 1$, then there exists a convex sum $pf + qg : A \to TB$. Moreover, for any $h : C \to TA, e : B \to TC$, we have the following:

$$\mu \circ T(pf + qg) \circ h = p(\mu \circ Tf \circ h) + q(\mu \circ Tg \circ h),$$

$$\mu \circ Te \circ (pf + qg) = p(\mu \circ Te \circ f) + q(\mu \circ Te \circ g).$$

**f** There are fully faithful embeddings $\mathbf{M} \overset{\psi}{\hookrightarrow} V(\mathbf{A})$ and $\mathbf{Q} \overset{\phi}{\hookrightarrow} Kl_{VT}(V(\mathbf{A}))$. These embedding functors are strong monoidal, and $\phi$ preserves the convex sum. Moreover, the following diagram commutes for any $S, U \in \mathbf{M}$.

$$\begin{tikzcd}
\mathbf{M}(S,U) \arrow[r, "\psi_{S,U}"] \arrow[d, "J_{S,U}"] & V(\mathbf{A})(S,U) \arrow[d, "E_{S,U}"] \\
\mathbf{Q}(S,U) \arrow[r, "\phi_{S,U}"] & Kl_{VT}(V(\mathbf{A}))(S,U)
\end{tikzcd}$$

Here, $E : V(\mathbf{A}) \to Kl_{VT}(V(\mathbf{A}))$ is the functor such that $E(A) = A$ and $E(f) = \eta \circ f$.

**g** Let $\mathscr{S}$ denote the set of objects in the image of $\psi$. For any $S, U \in \mathscr{S}$, there is an isomorphism

$$\flat(S \multimap U) \overset{e}{\cong} \mathbf{A}(S,U).$$

**h** There are maps $\mathsf{dynlift} : \mathbf{Bit} \to T\mathbf{Bool}$ and $\mathsf{init} : \mathbf{Bool} \to \mathbf{Bit}$ in **A** such that the following diagram commutes.

$$\begin{tikzcd}
& \mathbf{Bit} \arrow[d, "\mathsf{dynlift}"] \\
\mathbf{Bool} \arrow[ur, "\mathsf{init}"] \arrow[r, "\eta"] & T\mathbf{Bool}
\end{tikzcd}$$

## Remarks

- Condition **c** gives rise to a comonoid structure $\mathsf{dup}_X : pX \to pX \otimes pX$ and $\mathsf{discard}_X : pX \to I$ for any $X \in \mathscr{V}$. Moreover, for any map $f : X \to Y$ in $\mathscr{V}$, we have the following in **A**.

$$\mathsf{dup}_Y \circ pf = (pf \otimes pf) \circ \mathsf{dup}_X.$$

- Objects in the image of the functor $p$ are called *parameter objects* in **A**. Such objects are equipped with maps $\mathsf{dup} : A \to A \otimes A$ and $\mathsf{discard} : A \to I$. In particular, $\mathbf{Bool} := I + I = p(1) + p(1) = p(2)$ is a parameter object.

- Using condition **g**, we define $\mathsf{box} = p(e)$ and $\mathsf{unbox} = p(e^{-1})$, and we have

$$p\flat(S \multimap U) \overset{\mathsf{box}/\mathsf{unbox}}{\cong} p\mathbf{A}(S,U).$$

- Note that

$$Kl_{VT}(V(\mathbf{A}))(A,B) = V(\mathbf{A})(A,VTB) = \mathscr{V}(1,\mathbf{A}(A,TB)) = \mathscr{V}(1,Kl_T(\mathbf{A})(A,B)) = V(Kl_T(\mathbf{A}))(A,B).$$

- The Kleisli category $Kl_{VT}(V(\mathbf{A}))$ is monoidal because $VT$ is a commutative strong monad and $V(\mathbf{A})$ is monoidal. For any $f : A_1 \to VTB_1$ and $g : A_2 \to VTB_2$ in $Kl_{VT}(V(\mathbf{A}))$, we define $f \otimes g \in Kl_{VT}(V(\mathbf{A}))(A_1 \otimes A_2, B_1 \otimes B_2)$ by

$$A_1 \otimes A_2 \xrightarrow{f\otimes g} VTB_1 \otimes VTB_2 \xrightarrow{s} VT(B_1 \otimes VTB_2) \xrightarrow{Tt} VTVT(B_1 \otimes B_2) \xrightarrow{\mu} VT(B_1 \otimes B_2).$$

- Since $\psi(S) = \phi(S)$ for any $S \in \mathbf{M},\mathbf{Q}$, we define $\mathbf{Bit} = \psi(\mathbf{Bit}) = \phi(\mathbf{Bit}) \in \mathbf{A}$.
- Condition **f** expresses the requirement that the enriched category $\mathbf{A}$ must combine both categories $\mathbf{M}$ and $\mathbf{Q}$, i.e., they are subcategories of $V(\mathbf{A})$ and its Kleisli category, respectively. Thus $\mathbf{A}$ has both quantum circuits and quantum operations. The commutative diagram implies that a circuit in $\mathbf{A}$ can be used as a quantum operation.
- In [7], we have shown that conditions **a-h** are sufficient to give a model of Proto-Quipper-Dyn that is sound with respect to its type system and an operational semantics.

# 3 A biset-enriched category C and its category of presheaves $\overline{\mathbf{C}}$

## 3.1 Biset enrichment

We now begin our construction of a concrete model satisfying Definition 2.2. Let $\mathbf{2}$ be the category with two objects $0,1$ and one nontrivial arrow $0 \to 1$. Let $\mathscr{V} = \mathbf{Set}^{\mathbf{2}^{\mathbf{op}}}$ be the category of functors from $\mathbf{2}^{\mathbf{op}}$ to $\mathbf{Set}$. Concretely, the objects of $\mathscr{V}$ are triples $(A_0,A_1,f)$, where $A_0,A_1$ are sets and $f$ is a function $A_1 \to A_0$. We call such a triple a *biset*. A morphism in $\mathscr{V}$ from $(A_0,A_1,f)$ to $(B_0,B_1,g)$ is a pair $(h_0,h_1)$, where $h_0 : A_0 \to B_0$ and $h_1 : A_1 \to B_1$ are functions such that the following diagram commutes.

$$\begin{array}{ccc} A_1 & \xrightarrow{h_1} & B_1 \\ \downarrow f & & \downarrow g \\ A_0 & \xrightarrow{h_0} & B_0 \end{array}$$

Because it is a presheaf category, the category of bisets $\mathscr{V} = \mathbf{Set}^{\mathbf{2}^{\mathbf{op}}}$ is complete, cocomplete, and cartesian closed. We write $A \Rightarrow B$ to denote an exponential object in $\mathscr{V}$.

The category $\mathscr{V}$ is itself a $\mathscr{V}$-category where the hom-object $\mathscr{V}(A,B)$ is given by the exponential object $A \Rightarrow B$. We write $\mathrm{Hom}_{\mathscr{V}}(A,B)$ to denote a hom-set when viewing $\mathscr{V}$ as an ordinary category. Any set $X$ can be viewed as a trivial biset $(X,X,\mathrm{Id})$. Therefore, any ordinary category can be viewed as a trivial biset-enriched category. For example, $\mathbf{Set}$ can be viewed as a $\mathscr{V}$-category, where the hom-objects are given by $\mathbf{Set}(A,B) = (\mathbf{Set}(A,B),\mathbf{Set}(A,B),\mathrm{Id})$ for any $A,B \in \mathbf{Set}$.

**Definition 3.1.** We define $\mathscr{V}$-functors $U_0(A_0,A_1,a) = A_0 : \mathscr{V} \to \mathbf{Set}$, and $\Delta(X) = (X,X,\mathrm{Id}) : \mathbf{Set} \to \mathscr{V}$.

The $\mathscr{V}$-functor $\Delta$ is fully faithful and $U_0$ is strong monoidal. Note that there is also another functor $U_1(A_0,A_1,a) = A_1 : \mathscr{V} \to \mathbf{Set}$, but it is only an ordinary functor, not a $\mathscr{V}$-functor. This is because for $A,B \in \mathscr{V}$, there does not in general exist a morphism $A \Rightarrow B \to \mathbf{Set}(A_1,B_1)$ in $\mathscr{V}$. The functor $U_1$ will play no role in this paper, but the two $\mathscr{V}$-functors $U_0$ and $\Delta$ will be important.

**Proposition 3.2.** *There is a $\mathscr{V}$-adjunction $U_0 : \mathscr{V} \to \mathbf{Set} \dashv \Delta : \mathbf{Set} \to \mathscr{V}$. We write $T$ for the $\mathscr{V}$-monad $\Delta \circ U_0$, it is a commutative strong $\mathscr{V}$-monad.*

### 3.2 The $\mathscr{V}$-category C

In the following we define a non-trivial $\mathscr{V}$-category $\mathbf{C}$.

**Definition 3.3.** We define the $\mathscr{V}$-category $\mathbf{C}$ as following.

- The objects of $\mathbf{C}$ are the same as those of $\mathbf{M}$ and $\mathbf{Q}$.
- For objects $A,B \in \mathbf{C}$, we define $\mathbf{C}(A,B)$ as the following object of $\mathscr{V}$,

$$\mathbf{C}(A,B) = (\mathbf{Q}(A,B), \mathbf{M}(A,B), J_{AB} : \mathbf{M}(A,B) \to \mathbf{Q}(A,B)),$$

  where $J : \mathbf{M} \to \mathbf{Q}$ is the interpretation functor.
- For every object $A \in \mathbf{C}$, we have a morphism $u_A = (\tilde{\mathrm{Id}}_0, \tilde{\mathrm{Id}}_1) : 1 \to \mathbf{C}(A,A)$ in $\mathscr{V}$, where $\tilde{\mathrm{Id}}_1(*) = \mathrm{Id}_A : A \to A$ in $\mathbf{M}$ and $\tilde{\mathrm{Id}}_0(*) = \mathrm{Id}_A : A \to A$ in $\mathbf{Q}$.
- For any $A,B,C \in \mathbf{C}$, we have a morphism $c_{A,B,C} = (c_0, c_1) : \mathbf{C}(A,B) \times \mathbf{C}(B,C) \to \mathbf{C}(A,C)$ in $\mathscr{V}$, where $c_0 : \mathbf{Q}(A,B) \times \mathbf{Q}(B,C) \to \mathbf{Q}(A,C)$ and $c_1 : \mathbf{M}(A,B) \times \mathbf{M}(B,C) \to \mathbf{M}(A,C)$ are the compositions in $\mathbf{Q}$ and $\mathbf{M}$, respectively.

### 3.3 The $\mathscr{V}$-category $\overline{\mathbf{C}}$

The biset-enriched category $\mathbf{C}$ is symmetric monoidal. However, it is not closed. For that, we will need to work in the $\mathscr{V}$-enriched presheaf category $\overline{\mathbf{C}}$.

**Definition 3.4.** We define the $\mathscr{V}$-category $\overline{\mathbf{C}} = \mathscr{V}^{\mathbf{C}^{\mathrm{op}}}$. Concretely, an object $F \in \overline{\mathbf{C}}$ is a $\mathscr{V}$-functor $\mathbf{C}^{\mathrm{op}} \to \mathscr{V}$. Because $\mathscr{V}$ is complete, for any $F,G \in \overline{\mathbf{C}}$, we have a hom-object $\overline{\mathbf{C}}(F,G) \in \mathscr{V}$ that represents $\mathscr{V}$-natural transformations $F \to G$.

An object in $\overline{\mathbf{C}}$ is a $\mathscr{V}$-functor $F : \mathbf{C}^{\mathrm{op}} \to \mathscr{V}$. This means that for each $A \in \mathbf{C}^{\mathrm{op}}$, there is an object $FA \in \mathscr{V}$. And for any $A,B \in \mathbf{C}^{\mathrm{op}}$ there is a morphism $F_{AB} : \mathbf{C}^{\mathrm{op}}(A,B) \to FA \Rightarrow FB$ in $\mathscr{V}$, which is the following commutative diagram.

$$\begin{array}{ccc}
\mathbf{M}(B,A) & \xrightarrow{F^1_{AB}} & (FA \Rightarrow FB)_1 = \mathrm{Hom}_{\mathscr{V}}(FA,FB) \\
\big\downarrow{\scriptstyle J_{B,A}} & & \big\downarrow{\scriptstyle p_0} \\
\mathbf{Q}(B,A) & \xrightarrow{F^0_{AB}} & (FA \Rightarrow FB)_0 = \mathbf{Set}((FA)_0,(FB)_0).
\end{array}$$

Note that an element $h \in \mathrm{Hom}_{\mathscr{V}}(FA,FB)$ is a pair of function $(h_0,h_1)$ such that the following commutes.

$$\begin{array}{ccc}
(FA)_1 & \xrightarrow{h_1} & (FB)_1 \\
\big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\
(FA)_0 & \xrightarrow{h_0} & (FB)_0
\end{array}$$

Thus we define $p_0(h_0,h_1) = h_0$. So a $\mathscr{V}$-functor $F : \mathbf{C}^{\mathrm{op}} \to \mathscr{V}$ induces an ordinary functor $F^0 : \mathbf{Q}^{\mathrm{op}} \to \mathbf{Set}$, where $F^0(A) = (FA)_0$ and the function $\mathbf{Q}(B,A) \to \mathbf{Set}(F^0A, F^0B)$ is given by $F^0_{AB}$ for any $A,B \in \mathbf{Q}$.

**Proposition 3.5.** *The $\mathscr{V}$-category $\overline{\mathbf{C}}$ is a $\mathscr{V}$-monoidal closed category, where the tensor product $\otimes_{\mathrm{Day}}$ and linear exponential $\multimap_{\mathrm{Day}}$ are given by Day's convolution [3]. The tensor unit is defined by $I := yI = \mathbf{C}(-,I)$, where y is the $\mathscr{V}$-enriched Yoneda embedding functor.*

The $\mathscr{V}$-category $\overline{\mathbf{C}}$ has coproducts. Day's construction implies that the Day tensor product distributes over the coproducts, and that the $\mathscr{V}$-enriched Yoneda embedding $y : \mathbf{C} \hookrightarrow \overline{\mathbf{C}}$ is strong monoidal.

The $\mathscr{V}$-adjunction $U_0 \dashv \Delta$ and the $\mathscr{V}$-monad $T$ can be lifted to $\overline{\mathbf{C}}$.

**Definition 3.6.** We define $\mathscr{V}$-functors $\overline{U_0}(F) := U_0 \circ F : \mathscr{V}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, $\overline{\Delta}(F) := \Delta \circ F : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \to \mathscr{V}^{\mathbf{C}^{\mathrm{op}}}$, and $\overline{T} := \overline{\Delta} \circ \overline{U_0} : \mathscr{V}^{\mathbf{C}^{\mathrm{op}}} \to \mathscr{V}^{\mathbf{C}^{\mathrm{op}}}$.

Note that $\overline{\Delta}$ is fully faithful and that $\overline{U_0}$ is strong monoidal.

**Proposition 3.7.** *There is a $\mathscr{V}$-adjunction $\overline{U_0} : \mathscr{V}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \dashv \overline{\Delta} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \to \mathscr{V}^{\mathbf{C}^{\mathrm{op}}}$.*

*Proof.* For any $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}, G \in \overline{\mathbf{C}}$, we need to show that $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\overline{U_0}F, G) \cong \mathscr{V}^{\mathbf{C}^{\mathrm{op}}}(F, \overline{\Delta}G)$ that is $\mathscr{V}$-natural in $F$ and $G$. This is true since the following isomorphisms follow from properties of *end*.

$$\begin{aligned} V^{\mathbf{C}^{\mathrm{op}}}(F, \overline{\Delta}G) &\cong \int_{A \in \mathbf{C}} \mathscr{V}(FA, \Delta GA) \cong \int_{A \in \mathbf{C}} \mathbf{Set}(U_0 FA, GA) \cong \int_{A \in \mathbf{C}} \mathscr{V}(\Delta U_0 FA, \Delta GA) \\ &\cong \mathscr{V}^{\mathbf{C}^{\mathrm{op}}}(\overline{\Delta U_0}F, \overline{\Delta}G) \cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\overline{U_0}F, G). \end{aligned}$$ □

**Proposition 3.8.** *The monad $\overline{T}$ is a commutative strong monad.*

Proposition 3.8 is a consequence of the following more general theorem, whose proof can be found in Appendix D.

**Theorem 3.9.** *Let $\mathscr{V}$ be a complete, cocomplete, symmetric monoidal closed category. Let $\mathbf{A}$ be a $\mathscr{V}$-category. If $T$ is a commutative strong $\mathscr{V}$-monad on $\mathscr{V}$, then $\overline{T}(F) = T \circ F$ is a commutative strong $\mathscr{V}$-monad on $\mathscr{V}^{\mathbf{A}^{\mathrm{op}}}$.*

Consider a $\mathscr{V}$-functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. For any $A, B \in \mathbf{C}$, $FA \in \mathbf{Set}$, and the map $F_{AB} : \mathbf{C}(B,A) \to \mathbf{Set}(FA, FB)$ is uniquely determined by the function $F^0_{AB} : \mathbf{Q}(B,A) \to \mathbf{Set}(FA, FB)$. So $F$ is uniquely determined by $F^0 : \mathbf{Q}^{\mathrm{op}} \to \mathbf{Set}$. In fact, the following theorem holds (the proof is in Appendix B).

**Theorem 3.10.** *We have* $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \cong \mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}$.

The following proposition shows the maps in the Kleisli category of $\overline{T}$ are essentially maps in $\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}$.

**Proposition 3.11.** *For any $F, G \in \overline{\mathbf{C}}$, we have*

$$\overline{\mathbf{C}}(F, \overline{T}G) = \overline{\mathbf{C}}(F, \overline{\Delta U_0}G) \cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\overline{U_0}F, \overline{U_0}G) \cong \mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}(F^0, G^0).$$

### 3.4 A linear-non-linear adjunction in $\overline{\mathbf{C}}$

Suppose $F \in \overline{\mathbf{C}}$ and $V \in \mathscr{V}$. By definition, the *copower* $V \odot F$, if it exists, is an object $V \odot F \in \overline{\mathbf{C}}$ such that the isomorphism $\overline{\mathbf{C}}(V \odot F, G) \cong V \Rightarrow \overline{\mathbf{C}}(F, G)$ is $\mathscr{V}$-natural in $G \in \overline{\mathbf{C}}$.

**Definition 3.12.** Let $V \in \mathscr{V}, F \in \overline{\mathbf{C}}$. We define the copower $V \odot F$ in $\overline{\mathbf{C}}$ as follows:

$$(V \odot F)(A) = V \times FA : \mathbf{C}^{\mathrm{op}} \to \mathscr{V}.$$

The fact that the above is indeed a copower can be verified using the calculus of *ends*. For any $F, G \in \overline{\mathbf{C}}$, we have

$$\begin{aligned} \overline{\mathbf{C}}(V \odot F, G) &\cong \int_{A \in \mathbf{C}} V \times FA \Rightarrow GA \cong \int_{A \in \mathbf{C}} V \Rightarrow (FA \Rightarrow GA) \\ &\cong V \Rightarrow \int_{A \in \mathbf{C}} (FA \Rightarrow GA) \cong V \Rightarrow \overline{\mathbf{C}}(F, G). \end{aligned}$$

**Definition 3.13.** We define $\mathscr{V}$-functors $p(X) = X \odot I : \mathscr{V} \to \overline{\mathbf{C}}$ and $\flat(F) = \overline{\mathbf{C}}(I, F) : \overline{\mathbf{C}} \to \mathscr{V}$.

The $\mathscr{V}$-functors $p$ and $\flat$ form a linear-non-linear adjunction in the sense of Benton [1].

**Theorem 3.14.** *We have a $\mathscr{V}$-adjunction $p \dashv \flat$. Moreover, $p$ is strong monoidal.*

*Proof.* We have $\overline{\mathbf{C}}(pX,G) \cong \overline{\mathbf{C}}(X \odot I, G) \cong X \Rightarrow \overline{\mathbf{C}}(I,G) \cong X \Rightarrow \flat(G)$. Moreover, $p$ is a strong monoidal $\mathscr{V}$-functor. We have $p(1) = 1 \odot yI \cong 1 \times \mathbf{C}(-,I) \cong yI$ and

$$\begin{aligned} p(X) \otimes_{\mathrm{Day}} p(Y) &= \int^{A,B} \mathbf{C}(-,A\otimes B) \times X \times yI(A) \times Y \times yI(B) \\ &\cong X \times Y \times \int^{A,B} \mathbf{C}(-,A\otimes B) \times yI(A) \times yI(B) \cong X \times Y \times yI = p(X \times Y). \end{aligned}$$ □

**Theorem 3.15.** *For any $S,U \in \mathbf{C}$, there is an isomorphism $\flat(yS \multimap_{\mathrm{Day}} yU) \cong \mathbf{C}(S,U)$.*

*Proof.* We have $\flat(yS \multimap_{\mathrm{Day}} yU) = \overline{\mathbf{C}}(I, yS \multimap_{\mathrm{Day}} yU) \cong \overline{\mathbf{C}}(yS, yU) \cong \mathbf{C}(S,U)$. □

Applying $p$ to the above isomorphism yields $p\flat(yS \multimap_{\mathrm{Day}} yU) \cong p\mathbf{C}(S,U)$. This isomorphism is called the *box/unbox* isomorphism in [16].

## 4 A reflective subcategory $\widetilde{\mathbf{C}}$ of $\overline{\mathbf{C}}$

The $\mathscr{V}$-category $\overline{\mathbf{C}}$ itself is not a model for Proto-Quipper with dynamic lifting. For example, it does not have a map $\mathbf{Bit} \to \overline{T}\mathbf{Bool}$ for dynamic lifting. Namely, we define $\mathbf{Bool} := yI + yI = \mathbf{C}(-,I) + \mathbf{C}(-,I)$ and $\mathbf{Bit} := y\mathrm{Bit} = \mathbf{C}(-,\mathrm{Bit}) \in \overline{\mathbf{C}}$, where $\mathrm{Bit} \in \mathbf{C}$. Note that $\mathrm{Bit} = I + I$ in $\mathbf{Q}$. Consider the following

$$\begin{aligned} \overline{\mathbf{C}}(\mathbf{Bit}, \overline{T}\mathbf{Bool}) &\cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\overline{U_0}\mathbf{Bit}, \overline{U_0}\mathbf{Bool}) \cong \mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}(\mathbf{Bit}^0, \mathbf{Bool}^0) \\ &\cong \mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}(\mathbf{Q}(-,\mathrm{Bit}), \mathbf{Q}(-,I) + \mathbf{Q}(-,I)) = \mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}(\mathbf{Q}(-,I+I), \mathbf{Q}(-,I) + \mathbf{Q}(-,I)). \end{aligned}$$

So a map in $\overline{\mathbf{C}}(\mathbf{Bit}, \overline{T}\mathbf{Bool})$ is the same as a natural transformation from $\mathbf{Q}(-,I+I)$ to $\mathbf{Q}(-,I) + \mathbf{Q}(-,I)$ in $\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}$. Moreover, for condition **h** to be satisfied, this natural transformation should be a left inverse of the canonical natural transformation $\mathbf{Q}(-,I) + \mathbf{Q}(-,I) \to \mathbf{Q}(-,I+I)$. On the other hand, by the Yoneda lemma, every natural transformation from $\mathbf{Q}(-,I+I)$ to $\mathbf{Q}(-,I) + \mathbf{Q}(-,I)$ either takes all of its values in the left component or in the right component of the disjoint union. Therefore, it can't be a left inverse to $\mathbf{Q}(-,I) + \mathbf{Q}(-,I) \to \mathbf{Q}(-,I+I)$. It follows that dynamic lifting cannot be interpreted in $\overline{\mathbf{C}}$. To fix this, we now consider a reflective subcategory of $\overline{\mathbf{C}}$.

**Definition 4.1.** A $\mathscr{V}$-functor $F : \mathbf{C}^{\mathrm{op}} \to \mathscr{V}$ is called *smooth* if $F^0 : \mathbf{Q}^{\mathrm{op}} \to \mathbf{Set}$ is a product-preserving functor, i.e., $F^0(A+B) \cong F^0A \times F^0B$ for any $A,B \in \mathbf{Q}$.

Observe that for any $A \in \mathbf{C}$, the $\mathscr{V}$-enriched Yoneda embedding $y$ of $A$, which is $\mathbf{C}(-,A)$, is smooth. Because $\mathbf{C}(-,A)^0 = \mathbf{Q}(-,A)$, and for any $B_1, B_2 \in \mathbf{Q}$, we have $\mathbf{Q}(B_1 + B_2, A) \cong \mathbf{Q}(B_1,A) \times \mathbf{Q}(B_2,A)$. Thus, the codomain of $y$ consists of smooth $\mathscr{V}$-functors.

**Definition 4.2.** We define $\widetilde{\mathbf{C}}$ to be the full $\mathscr{V}$-subcategory of smooth functors.

**Definition 4.3.** We define the *Lambek embedding* $\overline{y} : \mathbf{C} \to \widetilde{\mathbf{C}}$ to be the corestriction of the Yoneda embedding $y$, i.e., it is the unique $\mathscr{V}$-functor such that the following diagram commutes.

$$\begin{tikzcd} \mathbf{C} \arrow[d, "\overline{y}"'] \arrow[dr, "y"] & \\ \widetilde{\mathbf{C}} \arrow[r, hook, "i"] & \overline{\mathbf{C}} \end{tikzcd}$$

The details of the proof of the following theorem are in Appendix E.

**Theorem 4.4.** *The $\mathscr{V}$-category $\widetilde{\mathbf{C}}$ is a reflective $\mathscr{V}$-subcategory of $\overline{\mathbf{C}}$, i.e., the inclusion $\mathscr{V}$-functor $i : \widetilde{\mathbf{C}} \hookrightarrow \overline{\mathbf{C}}$ has a left adjoint $\tilde{L}$.*

Using results of Day [4, 5] (see also [15] for a more recent exposition), we can furthermore show that that $\widetilde{\mathbf{C}}$ is symmetric monoidal and $\tilde{L}$ is strong monoidal. See Appendix F for further details. We now give an explicit definition of the monoidal closed structure in $\widetilde{\mathbf{C}}$.

**Definition 4.5.** For any $F, G \in \widetilde{\mathbf{C}}$, we define the tensor product, internal hom and tensor unit in $\widetilde{\mathbf{C}}$ as $F \otimes_{\mathrm{Lam}} G := \tilde{L}(iF \otimes_{\mathrm{Day}} iG)$, $F \multimap_{\mathrm{Lam}} G := iF \multimap_{\mathrm{Day}} iG$, and $I := \bar{y}I = \mathbf{C}(-, I)$, respectively.

In the above definition, the linear exponential $F \multimap_{\mathrm{Lam}} G$ is well-defined because $iF \multimap_{\mathrm{Day}} iG$ is an object in $\widetilde{\mathbf{C}}$ (Theorem F.1).

**Theorem 4.6.** *The $\mathscr{V}$-category $\widetilde{\mathbf{C}}$ is symmetric monoidal closed. For any $F, G, H \in \widetilde{\mathbf{C}}$, there is a $\mathscr{V}$-natural isomorphism $\widetilde{\mathbf{C}}(F \otimes_{\mathrm{Lam}} G, H) \cong \widetilde{\mathbf{C}}(F, G \multimap_{\mathrm{Lam}} H)$.*

*Proof.* We have $\widetilde{\mathbf{C}}(F \otimes_{\mathrm{Lam}} G, H) \cong \widetilde{\mathbf{C}}(\tilde{L}(iF \otimes_{\mathrm{Day}} iG), H) \cong \overline{\mathbf{C}}(iF \otimes_{\mathrm{Day}} iG, iH) \cong \overline{\mathbf{C}}(iF, iG \multimap_{\mathrm{Day}} iH) \cong \widetilde{\mathbf{C}}(F, G \multimap_{\mathrm{Lam}} H)$. □

## 4.1 A linear-non-linear adjunction in $\widetilde{\mathbf{C}}$

The $\mathscr{V}$-category $\widetilde{\mathbf{C}}$ also admits a linear-non-linear adjunction and, as in $\overline{\mathbf{C}}$, there is a box/unbox isomorphism in $\widetilde{\mathbf{C}}$.

**Definition 4.7.** We define the $\mathscr{V}$-functors $\tilde{p}(X) = \tilde{L}(p(X)) : \mathscr{V} \to \widetilde{\mathbf{C}}$ and $\tilde{\flat}(F) = \flat(iF) : \widetilde{\mathbf{C}} \to \mathscr{V}$.

**Theorem 4.8.** *We have a $\mathscr{V}$-adjunction $\tilde{p} \dashv \tilde{\flat}$. Moreover, $\tilde{p}$ is strong monoidal.*

*Proof.* We have $\widetilde{\mathbf{C}}(\tilde{p}X, F) = \widetilde{\mathbf{C}}(\tilde{L}(p(X)), F) \cong \overline{\mathbf{C}}(pX, iF) \cong X \Rightarrow \flat(iF) \cong X \Rightarrow \tilde{\flat}(F)$. Moreover, $\tilde{p}$ is strong monoidal because both $\tilde{L}$ and $p$ are strong monoidal. □

**Theorem 4.9.** *For any $S, U \in \mathbf{C}$, we have $\mathbf{C}(S, U) \cong \tilde{\flat}(\bar{y}S \multimap_{\mathrm{Lam}} \bar{y}U)$.*

*Proof.* We have $\tilde{\flat}(\bar{y}S \multimap_{\mathrm{Lam}} \bar{y}U) = \flat(i(\bar{y}S \multimap_{\mathrm{Lam}} \bar{y}U)) = \overline{\mathbf{C}}(I, i(\bar{y}S \multimap_{\mathrm{Lam}} \bar{y}U)) \cong \widetilde{\mathbf{C}}(I, \bar{y}S \multimap_{\mathrm{Lam}} \bar{y}U) \cong \widetilde{\mathbf{C}}(\bar{y}S, \bar{y}U) \cong \mathbf{C}(S, U)$. □

## 4.2 A commutative strong monad on $\widetilde{\mathbf{C}}$

The $\mathscr{V}$-category $\widetilde{\mathbf{C}}$ has a commutative strong monad. In the following we write $[\mathscr{V}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}$ for $\widetilde{\mathbf{C}}$ and $\mathscr{V}^{\mathbf{C}^{\mathrm{op}}}$ for $\overline{\mathbf{C}}$. We write $[\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$ for the full subcategory of product-preserving functors of $\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}$. Consider the following diagram.

$$\begin{tikzcd}
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \arrow[r, shift left, "\overline{\Delta}"'] \arrow[d, shift right, "L"'] & \mathscr{V}^{\mathbf{C}^{\mathrm{op}}} \arrow[l, shift left, "\overline{U_0}"'] \arrow[d, shift right, "\tilde{L}"'] \\
\left[\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}\right]_{\mathrm{prod}} \arrow[u, shift right, "j"'] \arrow[r, shift left, "\overline{\Delta}'"'] & \left[\mathscr{V}^{\mathbf{C}^{\mathrm{op}}}\right]_{\mathrm{prod}} \arrow[u, shift right, "i"'] \arrow[l, shift left, "\overline{U_0}'"']
\end{tikzcd}.$$

We define the $\mathscr{V}$-functor $\overline{U_0}' : [\mathscr{V}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}} \to [\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}$ by restricting the domain of $\overline{U_0}$ to $[\mathscr{V}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}$. Here $[\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}$ is the full $\mathscr{V}$-subcategory of smooth $\mathscr{V}$-functors. Similarly, $\overline{\Delta}' : [\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}} \to [\mathscr{V}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}$ is a restriction of $\overline{\Delta}$. We have a monoidal adjunction $L \dashv j$, since $[\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}} \cong [\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}$, the full subcategory of product-preserving functors, is reflective in $\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}} \cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$. We write $\widetilde{T} = \overline{\Delta}' \circ \overline{U_0}'$. Observe that $\widetilde{T}$ is $\overline{T}$ with a restricted domain.

**Proposition 4.10.** *By definition, we have $i \circ \overline{\Delta}' \cong \overline{\Delta} \circ j$ and $j \circ \overline{U_0}' \cong \overline{U_0} \circ i$, therefore $i \circ \widetilde{T} \cong \overline{T} \circ i$. Moreover, $\overline{U_0}' \circ \tilde{L} \cong L \circ \overline{U_0}$.*

**Theorem 4.11.** *We have a $\mathscr{V}$-adjunction $\overline{U_0}' \dashv \overline{\Delta}' : [\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}} \to [\mathscr{V}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}$. And $\overline{U_0}'$ is strong monoidal.*

*Proof.* For any $X \in [\mathscr{V}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}, Y \in [\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}$, we have

$$\begin{aligned}\left[\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}\right]_{\mathrm{prod}}(\overline{U_0}'X, Y) &\cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(j\overline{U_0}'X, jY) \cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\overline{U_0}iX, jY) \\ &\cong \mathscr{V}^{\mathbf{C}^{\mathrm{op}}}(iX, \overline{\Delta}jY) \cong \mathscr{V}^{\mathbf{C}^{\mathrm{op}}}(iX, i\overline{\Delta}'Y) \cong \left[\mathscr{V}^{\mathbf{C}^{\mathrm{op}}}\right]_{\mathrm{prod}}(X, \overline{\Delta}'Y).\end{aligned}$$

The $\mathscr{V}$-functor $\overline{U_0}'$ is strong monoidal. For any $F, G \in [\mathscr{V}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}$, we have $\overline{U_0}'I \cong \overline{U_0}I \cong I$ and

$$\begin{aligned}\overline{U_0}'(F \otimes_{\mathrm{Lam}} G) &= \overline{U_0}'\tilde{L}(iF \otimes_{\mathrm{Day}} iG) \cong L\overline{U_0}(iF \otimes_{\mathrm{Day}} iG) \cong L(\overline{U_0}iF \otimes_{\mathrm{Day}} \overline{U_0}iG) \\ &\cong L(j\overline{U_0}'F \otimes_{\mathrm{Day}} j\overline{U_0}'G) = \overline{U_0}'F \otimes_{\mathrm{Lam}} \overline{U_0}'G.\end{aligned}$$

□

**Theorem 4.12.** *There is a $\mathscr{V}$-natural transformation $\rho : \tilde{L} \circ \overline{T} \to \widetilde{T} \circ \tilde{L}$.*

*Proof.* For any $F \in \overline{\mathbf{C}}$, let $\eta_F : F \to i\tilde{L}F$ be the unit and $\varepsilon_F : \tilde{L}iF \to F$ be the counit (which is an isomorphism). We define $\rho_F$ to be the composition $\tilde{L}\overline{T}F \xrightarrow{\tilde{L}\overline{T}\eta_F} \tilde{L}\overline{T}i\tilde{L}F \xrightarrow{\cong} \tilde{L}i\widetilde{T}\tilde{L}F \xrightarrow{\varepsilon_{\widetilde{T}\tilde{L}F}} \widetilde{T}\tilde{L}F$. □

The natural transformation $\rho$ is one of the components for defining the strength for $\widetilde{T}$.

**Theorem 4.13.** *The $\mathscr{V}$-functor $\widetilde{T}$ is a commutative strong monad.*

*Proof.* For any $F, G \in \widetilde{\mathbf{C}}$, the strength of $\widetilde{T}$ is given by

$$F \otimes_{\mathrm{Lam}} \widetilde{T}G = \tilde{L}(iF \otimes_{\mathrm{Day}} i\widetilde{T}G) \xrightarrow{\cong} \tilde{L}(iF \otimes_{\mathrm{Day}} \overline{T}iG) \xrightarrow{\tilde{L}\bar{t}} \tilde{L}\overline{T}(iF \otimes_{\mathrm{Day}} iG) \xrightarrow{\rho} \widetilde{T}\tilde{L}(iF \otimes_{\mathrm{Day}} iG) = \widetilde{T}(F \otimes_{\mathrm{Lam}} G).$$

Note that $\bar{t}$ is the strength for $\overline{T}$. The verification of the strength diagrams is in Appendix G. □

Similarly to Proposition 3.11, we have the following theorem for $\widetilde{T}$.

**Theorem 4.14.** *For any $F, G \in \widetilde{\mathbf{C}}$, we have the following $\mathscr{V}$-natural isomorphisms.*

$$\widetilde{\mathbf{C}}(F, \widetilde{T}G) \cong [\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}(\overline{U_0}'F, \overline{U_0}'G) \cong [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}(F^0, G^0).$$

*Proof.* We have $\widetilde{\mathbf{C}}(F, \widetilde{T}G) = \widetilde{\mathbf{C}}(F, \overline{\Delta}'\overline{U_0}'G) \cong [\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}(\overline{U_0}'F, \overline{U_0}'G) \cong [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}(F^0, G^0)$. Note that by Theorem 3.10, $[\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}} \cong [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$. □

### 4.3 Dynamic lifting in $\widetilde{\mathbf{C}}$

Since $\overline{\mathbf{C}}$ has coproducts and $\widetilde{\mathbf{C}}$ is a reflective subcategory, the coproduct of $A, B \in \widetilde{\mathbf{C}}$ is defined as $A +' B = \tilde{L}(iA + iB)$. In $\widetilde{\mathbf{C}}$, we define $\mathbf{Bool} := \bar{y}I +' \bar{y}I = \tilde{L}(\bar{y}I + \bar{y}I)$ and $\mathbf{Bit} := \bar{y}(\mathrm{Bit})$, where $I, \mathrm{Bit} \in \mathbf{C}$. There exists maps $\mathsf{zero}, \mathsf{one} : \bar{y}I \to \mathbf{Bit}$ in $\widetilde{\mathbf{C}}$. We are now ready to define a map for dynamic lifting.

**Theorem 4.15.** *There are $\mathscr{V}$-natural transformations $\mathsf{init} : \mathbf{Bool} \to \mathbf{Bit}$ and $\mathsf{dynlift} : \mathbf{Bit} \to \widetilde{T}\mathbf{Bool}$ in $\widetilde{\mathbf{C}}$ such that the following diagram commutes.*

$$\begin{tikzcd} & \mathbf{Bit} \arrow[d, "\mathsf{dynlift}"] \\ \mathbf{Bool} \arrow[ur, "\mathsf{init}"] \arrow[r, "\eta"'] & \widetilde{T}\mathbf{Bool} \end{tikzcd}$$

*Proof.* We define $\mathsf{init} = [\mathsf{zero}, \mathsf{one}] : \mathbf{Bool} \to \mathbf{Bit}$. Firstly, we want to show that $\widetilde{T}\mathsf{init} : \widetilde{T}\mathbf{Bool} \to \widetilde{T}\mathbf{Bit}$ is an isomorphism. Using Yoneda's principle, we just need to show $\widetilde{\mathbf{C}}(F, \widetilde{T}\mathsf{init}) : \widetilde{\mathbf{C}}(F, \widetilde{T}\mathbf{Bool}) \to \widetilde{\mathbf{C}}(F, \widetilde{T}\mathbf{Bit})$ is an isomorphism for any $F \in \widetilde{\mathbf{C}}$. By Theorem 4.14, this is equivalent to showing that

$$[\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}(F^0, \mathsf{init}^0) : [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}(F^0, \mathbf{Bool}^0) \to [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}(F^0, \mathbf{Bit}^0)$$

is an isomorphism. This is the case because the Lambek embedding $\kappa : \mathbf{Q} \hookrightarrow [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$ preserves coproducts, $\mathrm{Bit} = I + I \in \mathbf{Q}$, and the map $\mathsf{init}^0 : \kappa I + \kappa I \to \kappa(I+I)$ is an isomorphism in $[\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$. We therefore define $\mathsf{dynlift}$ as the composition $(\widetilde{T}\mathsf{init})^{-1} \circ \eta : \mathbf{Bit} \to \widetilde{T}\mathbf{Bit} \to \widetilde{T}\mathbf{Bool}$. As a result, we have the following commutative diagram.

$$\begin{tikzcd}
\mathbf{Bool} \arrow[r, "\eta"] \arrow[d, "\mathsf{init}"] & \widetilde{T}\mathbf{Bool} \arrow[d, "\widetilde{T}\mathsf{init}"] \arrow[dr, "\mathrm{Id}"] & \\
\mathbf{Bit} \arrow[r, "\eta"] & \widetilde{T}\mathbf{Bit} \arrow[r, "(\widetilde{T}\mathsf{init})^{-1}"'] & \widetilde{T}\mathbf{Bool}
\end{tikzcd}$$

□

### 4.4 $\widetilde{\mathbf{C}}$ is a model for Proto-Quipper with dynamic lifting

Recall that the category $\mathbf{Q}$ is enriched in convex spaces, i.e., the hom-sets of $\mathbf{Q}$ are convex spaces and the composition is bilinear with respect to the convex sum. We have the following theorem, whose proof is in Appendix C.

**Theorem 4.16.** *The category $[\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$ is enriched in convex spaces. Moreover, the Lambek embedding $\kappa : \mathbf{Q} \hookrightarrow [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$ preserves the convex sum in $\mathbf{Q}$.*

The above theorem implies that for any $A, B \in \widetilde{\mathbf{C}}$, the Kleisli-hom $\widetilde{\mathbf{C}}(A, \widetilde{T}B)$ is convex because of the isomorphism $\widetilde{\mathbf{C}}(A, \widetilde{T}B) \cong [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}(A^0, B^0)$ from Theorem 4.14. We are now ready to state our main theorem (see Appendix H for the proof).

**Theorem 4.17.** *The $\mathscr{V}$-category $\widetilde{\mathbf{C}}$ is a model for Proto-Quipper with dynamic lifting, i.e., it satisfies conditions **a**–**h** in Definition 2.2.*

## 5 Conclusion

We constructed a categorical model for dynamic lifting using biset enrichment. We defined a biset-enriched category $\mathbf{C}$, which combines the categories $\mathbf{M}$ and $\mathbf{Q}$. We then considered the full subcategory $\widetilde{\mathbf{C}}$ of smooth functors and showed that $\widetilde{\mathbf{C}}$ is a reflective subcategory in the enriched presheaf category of $\mathbf{C}$. Finally, we proved that $\widetilde{\mathbf{C}}$ is categorical model for dynamic lifting in the sense of [7].

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# A Enriched symmetric monoidal categories

**Definition A.1.** Let $\mathscr{V}$ be a symmetric monoidal category. A $\mathscr{V}$-category $\mathbf{A}$ is symmetric monoidal if it is equipped with the following:

- There is an object $I$, called the *tensor unit*. For all $A,B \in \mathbf{A}$, there is an object $A \otimes B \in \mathbf{A}$. Moreover, for all $A_1,A_2,B_1,B_2 \in \mathbf{A}$, there is a map

$$\mathrm{Tensor} : \mathbf{A}(A_1,B_1) \otimes \mathbf{A}(A_2,B_2) \to \mathbf{A}(A_1 \otimes A_2, B_1 \otimes B_2)$$

  in $\mathscr{V}$. The tensor product is a bifunctor in the sense that $\mathrm{Tensor} \circ (u_A \otimes u_B) = u_{A\otimes B}$ for the identity maps $u_A, u_B, u_{A\otimes B}$, and the following diagram commutes for any $A_1,A_2,B_1,B_2,C_1,C_2 \in \mathbf{A}$.

$$\begin{tikzcd}
\mathbf{A}(A_1,B_1) \otimes \mathbf{A}(A_2,B_2) \otimes \mathbf{A}(B_1,C_1) \otimes \mathbf{A}(B_2,C_2) \arrow[r, "c\otimes c"] \arrow[d, "\mathrm{Tensor}\otimes\mathrm{Tensor}"] & \mathbf{A}(A_1,C_1) \otimes \mathbf{A}(A_2,C_2) \arrow[d, "\mathrm{Tensor}"] \\
\mathbf{A}(A_1 \otimes A_2, B_1 \otimes B_2) \otimes \mathbf{A}(B_1 \otimes B_2, C_1 \otimes C_2) \arrow[r, "c"] & \mathbf{A}(A_1 \otimes A_2, C_1 \otimes C_2)
\end{tikzcd}$$

- There are the following $\mathscr{V}$-natural isomorphisms in $\mathbf{A}$ and they satisfy the same coherence diagrams as for symmetric monoidal categories, and analogous naturality conditions.

$$l_A : I \otimes A \to A$$

$$r_A : A \otimes I \to A$$

$$\gamma_{A,B} : A \otimes B \to B \otimes A$$

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

If the $\mathscr{V}$-category $\mathbf{A}$ is symmetric monoidal, for all maps $f : A_1 \to B_1, g : A_2 \to B_2$ in $\mathbf{A}$, we write $f \otimes g : A_1 \otimes A_2 \to B_1 \otimes B_2$ as a shorthand for the following composition.

$$I \overset{f\otimes g}{\to} \mathbf{A}(A_1,B_1) \otimes \mathbf{A}(A_2,B_2) \overset{\mathrm{Tensor}}{\to} \mathbf{A}(A_1 \otimes A_2, B_1 \otimes B_2)$$

# B Biset-enriched functor categories

**Notations**. Let $\mathbf{A},\mathbf{B}$ be $\mathscr{V}$-categories. For all $A,B \in \mathbf{A}$, we have

$$\mathbf{A}(A,B) = (\mathbf{A}(A,B)_0, \mathbf{A}(A,B)_1, \varphi^{\mathbf{A}} : \mathbf{A}(A,B)_1 \to \mathbf{A}(A,B)_0).$$

So we write $A \to_1 B := \mathbf{A}(A,B)_1$ and $A \to_0 B := \mathbf{A}(A,B)_0$. Moreover, for all $f : A \to_1 B$, we have $\varphi^{\mathbf{A}}(f) : A \to_0 B$. A $\mathscr{V}$-functor $F : \mathbf{A} \to \mathbf{B}$ gives rise to the following commutative diagram for all $A,B \in \mathbf{A}$.

$$\begin{tikzcd}
\mathbf{A}(A,B)_1 \arrow[r, "F^1_{A,B}"] \arrow[d, "\varphi^{\mathbf{A}}"] & \mathbf{B}(FA,FB)_1 \arrow[d, "\varphi^{\mathbf{B}}"] \\
\mathbf{A}(A,B)_0 \arrow[r, "F^0_{A,B}"] & \mathbf{B}(FA,FB)_0
\end{tikzcd}$$

For all $f : A \to_1 B$, we have $F^1_{A,B} f : FA \to_1 FB$. Similarly, for all $g : A \to_0 B$, we have $F^0_{A,B} g : FA \to_0 FB$.

For any $\mathscr{V}$-functors $F, G : \mathbf{A} \to \mathbf{B}$, we define a biset $(F \Rightarrow_0 G, F \Rightarrow_1 G, p : F \Rightarrow_1 G \to F \Rightarrow_0 G)$ as follows.

$$F \Rightarrow_0 G := \{(\beta_A : FA \to_0 GA)_{A \in \mathbf{A}} \mid \forall A, B \in \mathbf{A}, \forall g : A \to_0 B, \beta_B \circ F^0_{AB} g = G^0_{AB} g \circ \beta_A\}$$

$$\begin{aligned} F \Rightarrow_1 G := \{(\alpha_A : FA \to_1 GA)_{A \in \mathbf{A}} \mid \forall A, B \in \mathbf{A}, \forall f : A \to_1 B, \alpha_B \circ F^1_{AB} f = G^1_{AB} f \circ \alpha_A, \\ \forall A, B \in \mathbf{A}, \forall g : A \to_0 B, \varphi^{\mathbf{B}}(\alpha_B) \circ F^0_{AB} g = G^0_{AB} g \circ \varphi^{\mathbf{B}}(\alpha_A)\} \end{aligned}$$

$$p((\alpha_A : FA \to_1 GA)_{A \in \mathbf{A}}) := (\varphi^{\mathbf{B}}(\alpha_A) : FA \to_0 GA)_{A \in \mathbf{A}} : F \Rightarrow_1 G \to F \Rightarrow_0 G$$

**Proposition B.1.** *Suppose $\mathbf{A}, \mathbf{B}$ are $\mathscr{V}$-categories. Since the category of bisets $\mathscr{V}$ is complete, the functor category $\mathbf{B}^{\mathbf{A}}$ is $\mathscr{V}$-enriched. For all $\mathscr{V}$-functors $F, G : \mathbf{A} \to \mathbf{B}$, we have*

$$\mathbf{B}^{\mathbf{A}}(F, G) := \int_{A \in \mathbf{A}} \mathbf{B}(FA, GA) \cong (F \Rightarrow_0 G, F \Rightarrow_1 G, p : F \Rightarrow_1 G \to F \Rightarrow_0 G)$$

*Proof.* By definition of end, we have the following equalizer diagram in $\mathscr{V}$.

$$\textstyle\int_{A \in \mathbf{A}} \mathbf{B}(FA, GA) := eq(u, v) \overset{k}{\rightarrowtail} \prod_{A \in \mathbf{A}} \mathbf{B}(FA, GA) \underset{v}{\overset{u}{\rightrightarrows}} \prod_{A, B \in \mathbf{A}} \mathbf{A}(A, B) \Rightarrow \mathbf{B}(FA, GB)$$

Note that $u = \langle \mathrm{curry}(c \circ (\pi_A \times G_{AB})) \rangle_{A, B \in \mathbf{A}}$, where $c \circ (\pi_A \times G_{AB})$ is the following.

$$(\textstyle\prod_A \mathbf{B}(FA, GA)) \times \mathbf{A}(A, B) \xrightarrow{\pi_A \times G_{AB}} \mathbf{B}(FA, GA) \times \mathbf{B}(GA, GB) \xrightarrow{\quad c \quad} \mathbf{B}(FA, GB)$$

We have $v = \langle \mathrm{curry}(c \circ (\pi_B \times F_{AB})) \rangle_{A, B \in \mathbf{A}}$, where $c \circ (\pi_B \times F_{AB})$ is the following.

$$(\textstyle\prod_A \mathbf{B}(FA, GA)) \times \mathbf{A}(A, B) \xrightarrow{\pi_B \times F_{AB}} \mathbf{B}(FB, GB) \times \mathbf{B}(FA, FB) \xrightarrow{\quad c \quad} \mathbf{B}(FA, GB)$$

We can show $(\int_{A \in \mathbf{A}} \mathbf{B}(FA, GA))_1 = eq(u_1, v_1) \cong F \Rightarrow_1 G$ and $(\int_{A \in \mathbf{A}} \mathbf{B}(FA, GA))_0 = eq(u_0, v_0) \cong F \Rightarrow_0 G$. $\square$

**Theorem B.2.** *The biset-enriched categories $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and $\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}$ are isomorphic.*

*Proof.* Let us define a $\mathscr{V}$-enriched functor $\Omega : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}$. On objects, $\Omega(F) = F^0$ for any $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$. Since $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is uniquely determined by $F^0$, the function $\Omega$ is bijective on objects.

Suppose $F, G : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. We claim that $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(F, G) \cong \mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}(F^0, G^0)$. This will allow us to define $\Omega_{F,G}$ to be this isomorphism. To show $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(F, G) \cong \mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}(F^0, G^0)$, first of all, we have

$$\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}(F^0, G^0) = (X, X, \mathrm{Id}),$$

where

$$X = \{(\alpha_A : F^0 A \to G^0 A)_{A \in \mathbf{Q}} \mid \forall A, B \in \mathbf{Q}, \forall f : A \to B \in \mathbf{Q}, \alpha_B \circ F^0_{AB} f = G^0_{AB} f \circ \alpha_A\}.$$

Next,

$$\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(F, G) = (F \Rightarrow_1 G, F \Rightarrow_0 G, p),$$

where

$$F \Rightarrow_0 G = \{(\alpha_A : FA \to_0 GA)_{A\in\mathbf{C}} \mid \forall A,B \in \mathbf{C}, \forall f : A \to_0 B \in \mathbf{C}, \alpha_B \circ F^0_{AB} f = G^0_{AB} f \circ \alpha_A\} \cong X$$

and

$$F \Rightarrow_1 G := \{(\alpha_A : FA \to_1 GA)_{A\in\mathbf{A}} \mid \forall A,B \in \mathbf{C}, \forall f : A \to_1 B, \alpha_B \circ F^1_{AB} f = G^1_{AB} f \circ \alpha_A,$$

$$\forall A,B \in \mathbf{C}, \forall g : A \to_0 B, \varphi^{\mathbf{Set}}(\alpha_B) \circ F^0_{AB} g = G^0_{AB} g \circ \varphi^{\mathbf{Set}}(\alpha_A)\}.$$

Since $F^1_{A,B} = F^0_{A,B} \circ \varphi^{\mathbf{C}^{\mathrm{op}}}$, and $\varphi^{\mathbf{Set}} = \mathrm{Id}$, and $\varphi^{\mathbf{C}^{\mathrm{op}}}(f) : A \to_0 B$ for any $f : A \to_1 B$ with $A,B \in \mathbf{C}$, therefore $\forall A,B \in \mathbf{C}, \forall g : A \to_0 B, \varphi^{\mathbf{Set}}(\alpha_B) \circ F^0_{AB} g = G^0_{AB} g \circ \varphi^{\mathbf{Set}}(\alpha_A)$ implies $\forall A,B \in \mathbf{C}, \forall f : A \to_1 B, \alpha_B \circ F^1_{AB} f = G^1_{AB} f \circ \alpha_A$. So $F \Rightarrow_1 G \cong F \Rightarrow_0 G \cong X$ and $p = \mathrm{Id}$. $\square$

## C Convexity

Let $[0,1]$ denote the real unit interval.

**Definition C.1.** A *convexity structure* on a set $X$ is an operation that assigns to all $p,q \in [0,1]$ with $p+q=1$ and all $x,y \in X$ an element $px+qy \in X$, subject to the following properties. Throughout, we assume $p+q=1$.

(a) $px+qx = x$ for all $x \in X$.

(b) $px+qy = qy+px$ for all $x,y \in X$.

(c) $0x+1y = y$ for all $x,y \in X$.

(d) $(a+b)(\frac{a}{a+b}x+\frac{b}{a+b}y)+(c+d)(\frac{c}{c+d}z+\frac{d}{c+d}w) = (a+c)(\frac{a}{a+c}x+\frac{c}{a+c}z)+(b+d)(\frac{b}{b+d}y+\frac{d}{b+d}w)$, where $a,b,c,d \in [0,1]$ with $a+b+c+d=1$ and all denominators are non-zero.

**Remark.** Property $(d)$ can best be understood by realizing that both sides of the equation are equal to $ax+by+cz+dw$, decomposed in two different ways into convex sums of two elements at a time. In the literature, we sometimes find a different, but equivalent condition of the form $s(px+qy)+rz = spx+(qs+r)(\frac{qs}{qs+r}y+\frac{r}{qs+r}z)$. The latter axiom is arguably shorter, but harder to read.

We often expand the binary $+$ operation to a multi-arity operation, i.e., $\sum_i p_i x_i$, where $\sum_i p_i = 1$ and $x_i \in X$ for all $i$.

We say that a category $\mathbf{A}$ is *enriched in convex spaces* if for all $A,B \in \mathbf{A}$, the hom-set $\mathbf{A}(A,B)$ is convex, and composition is bilinear, i.e., for all $f,g \in \mathbf{A}(A,B), e \in \mathbf{A}(C,A), h \in \mathbf{A}(B,C)$ and $p,q \in [0,1]$ with $p+q=1$, we have

$$(pf+qg)\circ e = pf\circ e + qg\circ e$$

and

$$h\circ(pf+qg) = ph\circ f + qh\circ g.$$

**Theorem C.2.** *Let $\mathbf{A}$ be a symmetric monoidal category with a coproduct $I+I$, such that tensor distributes over this coproduct. The following are equivalent.*

1. *The category $\mathbf{A}$ is enriched in convex spaces.*

2. *There exists a family of maps $\langle\!\langle p,q\rangle\!\rangle : I \to I+I$, where $p,q \in [0,1]$ with $p+q=1$, such that the following diagrams commute:*

$$
\begin{tikzcd}
I \arrow[r, "\langle\!\langle p,q\rangle\!\rangle"] \arrow[rr, bend right, "\mathrm{Id}"'] & I+I \arrow[r, "{[\mathrm{Id},\mathrm{Id}]}"] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\langle\!\langle p,q\rangle\!\rangle"] \arrow[dr, "\langle\!\langle q,p\rangle\!\rangle"'] & I+I \arrow[d, "{[\mathrm{inj}_2,\mathrm{inj}_1]}"] \\
& I+I
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, bend left, "\mathrm{inj}_2"] \arrow[r, bend right, "\langle\!\langle 0,1\rangle\!\rangle"'] & I+I
\end{tikzcd}
$$

$$
\begin{tikzcd}
& I \arrow[dl, "\langle\!\langle a+b,c+d\rangle\!\rangle"'] \arrow[dr, "\langle\!\langle a+c,b+d\rangle\!\rangle"] & \\
I+I \arrow[d, "\langle\!\langle \frac{a}{a+b},\frac{b}{a+b}\rangle\!\rangle+\langle\!\langle \frac{c}{c+d},\frac{d}{c+d}\rangle\!\rangle"'] & & I+I \arrow[d, "\langle\!\langle \frac{a}{a+c},\frac{c}{a+c}\rangle\!\rangle+\langle\!\langle \frac{b}{b+d},\frac{d}{b+d}\rangle\!\rangle"] \\
(I+I)+(I+I) \arrow[rr, "\mathrm{iso}"] & & (I+I)+(I+I)
\end{tikzcd}
$$

*Here, in the last diagram, we have $a,b,c,d \in [0,1]$ with $a+b+c+d=1$, and we assume the denominators are non-zero. The map "iso" is the canonical isomorphism $(A+B)+(C+D) \cong (A+C)+(B+D)$.*

*Proof.* For the left-to-right implication, suppose $\mathbf{A}$ is enriched in convex spaces. We can define

$$\langle\!\langle p,q\rangle\!\rangle := p\,\mathrm{inj}_1 + q\,\mathrm{inj}_2 : I \to I+I.$$

It is easy to verify that this definition of $\langle\!\langle p,q\rangle\!\rangle$ satisfies the four diagrams above.

We now focus on the right-to-left implication.

- First we need to show that $\mathbf{A}(A,B)$ is convex for all $A,B \in \mathbf{A}$. Given $f,g \in \mathbf{A}(A,B)$, we define $pf+qg$ as follows.

$$A \xrightarrow{\lambda^{-1}} A\otimes I \xrightarrow{A\otimes\langle\!\langle p,q\rangle\!\rangle} A\otimes(I+I) \xrightarrow{d} A\otimes I + A\otimes I \xrightarrow{\lambda+\lambda} A+A \xrightarrow{[f,g]} B.$$

- $pf+qf=f$. This holds because the following diagram commutes.

$$
\begin{tikzcd}
A \arrow[r, "\lambda^{-1}"] \arrow[d, "f"] & A\otimes I \arrow[r, "A\otimes\langle\!\langle p,q\rangle\!\rangle"] \arrow[d, "f\otimes I"] & A\otimes(I+I) \arrow[r, "d"] \arrow[d, "f\otimes(I+I)"] & A\otimes I+A\otimes I \arrow[r, "\lambda+\lambda"] \arrow[d, "f\otimes I+f\otimes I"] & A+A \arrow[r, "{[f,f]}"] \arrow[d, "f+f"'] & B \\
B \arrow[r, "\lambda^{-1}"] & B\otimes I \arrow[r, "B\otimes\langle\!\langle p,q\rangle\!\rangle"] \arrow[dr, "\mathrm{Id}"] & B\otimes(I+I) \arrow[r, "d"] \arrow[d, "{B\otimes[\mathrm{Id},\mathrm{Id}]}"] & B\otimes I+B\otimes I \arrow[r, "\lambda+\lambda"] \arrow[dl, bend left, "{[\mathrm{Id},\mathrm{Id}]}"] & B+B \arrow[ur, "{[\mathrm{Id},\mathrm{Id}]}"] & \\
& & B\otimes I \arrow[uurrr, bend right, "\lambda"'] & & &
\end{tikzcd}
$$

- $pf+qg=qg+pf$. This holds because the following diagram commutes.

$$
\begin{tikzcd}
A \arrow[r, "\lambda^{-1}"] & A\otimes I \arrow[r, "A\otimes\langle\!\langle p,q\rangle\!\rangle"] \arrow[dr, "A\otimes\langle\!\langle q,p\rangle\!\rangle"'] & A\otimes(I+I) \arrow[r, "d"] \arrow[d, "{A\otimes[\mathrm{inj}_2,\mathrm{inj}_1]}"] & A\otimes I+A\otimes I \arrow[r, "\lambda+\lambda"] & A+A \arrow[r, "{[f,g]}"] \arrow[d, "{[\mathrm{inj}_2,\mathrm{inj}_1]}"'] & B \\
& & A\otimes(I+I) \arrow[r, "d"] & A\otimes I+A\otimes I \arrow[r, "\lambda+\lambda"] & A+A \arrow[ur, "{[g,f]}"'] &
\end{tikzcd}
$$

- $0f+1g=g$. We have the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "\lambda^{-1}"] \arrow[dddrr, "\mathrm{inj}_2"] \arrow[ddddrr, bend right, "g"'] & A\otimes I \arrow[dr, bend left, "A\otimes\langle\!\langle 0,1\rangle\!\rangle"] \arrow[dr, "A\otimes\mathrm{inj}_2"'] \arrow[ddr, bend right, "\mathrm{inj}_2"] & \\
& & A\otimes(I+I) \arrow[d, "d"] \\
& & A\otimes I + A\otimes I \arrow[d, "\lambda+\lambda"] \\
& & A+A \arrow[d, "{[f,g]}"] \\
& & B
\end{tikzcd}$$

- $(a+b)(\frac{a}{a+b}f+\frac{b}{a+b}g)+(c+d)(\frac{c}{c+d}h+\frac{d}{c+d}w)=(a+c)(\frac{a}{a+c}f+\frac{c}{a+c}h)+(b+d)(\frac{b}{b+d}g+\frac{d}{b+d}w)$.
Let us write $\alpha=\frac{a}{a+b}f+\frac{b}{a+b}g$ and $\beta=\frac{c}{c+d}h+\frac{d}{c+d}w$. We have the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "\lambda^{-1}"] & A\otimes I \arrow[r, "A\otimes\langle\!\langle a+b,c+d\rangle\!\rangle"] & A\otimes(I+I) \arrow[r, "d"] \arrow[d, "A\otimes(\langle\!\langle \frac{a}{a+b},\frac{b}{a+b}\rangle\!\rangle+\langle\!\langle \frac{c}{c+d},\frac{d}{c+d}\rangle\!\rangle)"'] & A\otimes I+A\otimes I \arrow[r, "\lambda+\lambda"] \arrow[ddl, "A\otimes\langle\!\langle \frac{a}{a+b},\frac{b}{a+b}\rangle\!\rangle+A\otimes\langle\!\langle \frac{c}{c+d},\frac{d}{c+d}\rangle\!\rangle"] & A+A \arrow[r, "{[\alpha,\beta]}"] \arrow[ddddl, "\alpha+\beta"'] & B \\
& & A\otimes((I+I)+(I+I)) \arrow[d, "d"'] & & & \\
& & A\otimes(I+I)+A\otimes(I+I) \arrow[d, "d+d"'] & & & \\
& & (A\otimes I+A\otimes I)+(A\otimes I+A\otimes I) \arrow[d, "(\lambda+\lambda)+(\lambda+\lambda)"'] & & & \\
& & (A+A)+(A+A) \arrow[r, "{[f,g]+[h,w]}"] & B+B \arrow[uuuurr, bend right, "{[\mathrm{Id},\mathrm{Id}]}"'] & &
\end{tikzcd}$$

Thus

$$\begin{aligned}
&(a+b)(\tfrac{a}{a+b}f+\tfrac{b}{a+b}g)+(c+d)(\tfrac{c}{c+d}h+\tfrac{d}{c+d}w)\\
&\quad=[\mathrm{Id},\mathrm{Id}]\circ([f,g]+[h,w])\circ((\lambda+\lambda)+(\lambda+\lambda))\circ(d+d)\circ d\\
&\qquad\circ(A\otimes(\langle\!\langle\tfrac{a}{a+b},\tfrac{b}{a+b}\rangle\!\rangle+\langle\!\langle\tfrac{c}{c+d},\tfrac{d}{c+d}\rangle\!\rangle))\circ(A\otimes\langle\!\langle a+b,c+d\rangle\!\rangle)\circ\lambda^{-1}.
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
&(a+c)(\tfrac{a}{a+c}f+\tfrac{c}{a+c}h)+(b+d)(\tfrac{b}{b+d}g+\tfrac{d}{b+d}w)\\
&\quad=[\mathrm{Id},\mathrm{Id}]\circ([f,h]+[g,w])\circ((\lambda+\lambda)+(\lambda+\lambda))\circ(d+d)\circ d\\
&\qquad\circ(A\otimes(\langle\!\langle\tfrac{a}{a+c},\tfrac{c}{a+c}\rangle\!\rangle+\langle\!\langle\tfrac{b}{b+d},\tfrac{d}{b+d}\rangle\!\rangle))\circ(A\otimes\langle\!\langle a+c,b+d\rangle\!\rangle)\circ\lambda^{-1}.
\end{aligned}$$

Thus we can show

$$(a+b)(\tfrac{a}{a+b}f+\tfrac{b}{a+b}g)+(c+d)(\tfrac{c}{c+d}h+\tfrac{d}{c+d}w)=(a+c)(\tfrac{a}{a+c}f+\tfrac{c}{a+c}h)+(b+d)(\tfrac{b}{b+d}g+\tfrac{d}{b+d}w)$$

by the following commutative diagram.

$$
\begin{tikzcd}
 & A\otimes I \arrow[dl, "A\otimes\langle\!\langle a+b,c+d\rangle\!\rangle"'] \arrow[dr, "A\otimes\langle\!\langle a+c,b+d\rangle\!\rangle"] & \\
A\otimes(I+I) \arrow[d, "A\otimes(\langle\!\langle \frac{a}{a+b},\frac{b}{a+b}\rangle\!\rangle+\langle\!\langle \frac{c}{c+d},\frac{d}{c+d}\rangle\!\rangle)"] & & A\otimes(I+I) \arrow[d, "A\otimes(\langle\!\langle \frac{a}{a+c},\frac{c}{a+c}\rangle\!\rangle+\langle\!\langle \frac{b}{b+d},\frac{d}{b+d}\rangle\!\rangle)"] \\
A\otimes((I+I)+(I+I)) \arrow[rr, "A\otimes\mathrm{iso}"] \arrow[d, "d"] & & A\otimes((I+I)+(I+I)) \arrow[d, "d"] \\
A\otimes(I+I)+A\otimes(I+I) \arrow[d, "d+d"] & & A\otimes(I+I)+A\otimes(I+I) \arrow[d, "d+d"] \\
(A\otimes I+A\otimes I)+(A\otimes I+A\otimes I) \arrow[rr, "\mathsf{iso}"] \arrow[d, "(\lambda+\lambda)+(\lambda+\lambda)"] & & (A\otimes I+A\otimes I)+(A\otimes I+A\otimes I) \arrow[d, "(\lambda+\lambda)+(\lambda+\lambda)"] \\
(A+A)+(A+A) \arrow[rr, "\mathsf{iso}"] \arrow[d, "{[f,g]+[h,w]}"] & & (A+A)+(A+A) \arrow[d, "{[f,w]+[g,h]}"] \\
B+B \arrow[r, "{[\mathrm{Id},\mathrm{Id}]}"] & B & B+B \arrow[l, "{[\mathrm{Id},\mathrm{Id}]}"']
\end{tikzcd}
$$

- $(pf+qg)\circ e = p(f\circ e)+q(g\circ e)$. This is by the following commutative diagram.

$$
\begin{tikzcd}
C \arrow[r, "e"] \arrow[drr, "\lambda^{-1}"'] & A \arrow[r, "\lambda^{-1}"] & A\otimes I \arrow[r, "A\otimes\langle\!\langle p,q\rangle\!\rangle"] & A\otimes(I+I) \arrow[r, "d"] & A\otimes I+A\otimes I \arrow[r, "\lambda+\lambda"] & A+A \arrow[r, "{[f,g]}"] & B \\
 & & C\otimes I \arrow[u, "e\otimes I"] \arrow[r, "C\otimes\langle\!\langle p,q\rangle\!\rangle"] & C\otimes(I+I) \arrow[u, "e\otimes(I+I)"] \arrow[r, "d"] & C\otimes I+C\otimes I \arrow[u, "e\otimes I+e\otimes I"] \arrow[r, "\lambda+\lambda"] & C+C \arrow[u, "e+e"] \arrow[ur, "{[f\circ e,g\circ e]}"'] &
\end{tikzcd}
$$

- $h\circ(pf+qg) = p(h\circ f)+q(h\circ g)$. This is by the following.

$$
\begin{tikzcd}
A \arrow[r, "\lambda^{-1}"] & A\otimes I \arrow[r, "A\otimes\langle\!\langle p,q\rangle\!\rangle"] & A\otimes(I+I) \arrow[r, "d"] & A\otimes I+A\otimes I \arrow[r, "\lambda+\lambda"] & A+A \arrow[r, "{[f,g]}"] \arrow[rr, bend right, "{[h\circ f,h\circ g]}"'] & B \arrow[r, "h"] & C
\end{tikzcd}
$$

□

**Theorem C.3.** *The category* $[\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$ *is enriched in convex spaces. Moreover, the Lambek embedding* $\kappa : \mathbf{Q} \hookrightarrow [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$ *preserves the convex sum in* $\mathbf{Q}$*.*

*Proof.* By Theorem C.2 (2), there exists a map $\langle\!\langle p,q\rangle\!\rangle : I \to I+I$ in $\mathbf{Q}$ for any $p,q\in[0,1]$, $p+q=1$, and it satisfies the four diagrams. Since $\kappa$ preserves coproducts in $\mathbf{Q}$, the map $\kappa\langle\!\langle p,q\rangle\!\rangle : \kappa I \to \kappa I +' \kappa I$ in $[\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$ also satisfies the four diagrams in Theorem C.2 (2). Therefore $[\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$ is enriched in convex spaces.

For all $f,g\in\mathbf{Q}(A,B)$, the convex sum $pf+qg\in\mathbf{Q}(A,B)$ is defined to be the following.

$$A \xrightarrow{\lambda^{-1}} A\otimes I \xrightarrow{A\otimes\langle\!\langle p,q\rangle\!\rangle} A\otimes(I+I) \xrightarrow{d} A\otimes I+A\otimes I \xrightarrow{\lambda+\lambda} A+A \xrightarrow{[f,g]} B$$

Since $\kappa$ preserves coproducts in $\mathbf{Q}$ and it is strong monoidal, we have $\kappa(pf+qg) = p\kappa(f)+q\kappa(g) \in [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}(\kappa A,\kappa B)$.

□

# D Proof of Theorem 3.9

In this section, we assume $\mathscr{V}$ to be a complete, cocomplete, symmetric monoidal closed category. The following proposition is due to Kock [13].

**Proposition D.1.** *Let $T : \mathscr{V} \to \mathscr{V}$ be a $\mathscr{V}$-monad. Then $T$ is a strong monad with strength $t : A \otimes TB \to T(A \otimes B)$ given by the following commutative diagram. Note that $\eta$ is the unit of the adjunction $- \otimes A \dashv A \multimap -$.*

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{curry}(t)"] \arrow[d, "\eta"] & & TB \multimap T(A \otimes B) \\
B \multimap A \otimes B \arrow[urr, "T_{B,A\otimes B}"'] & &
\end{tikzcd}$$

**Theorem D.2.** *Let $T$ be a strong monad on $\mathscr{V}$ and $F : \mathbf{A}^{\mathrm{op}} \to \mathscr{V}$ be a $\mathscr{V}$-functor. For all $A, B \in \mathbf{A}$, we have maps*

$$F_{AB} : \mathbf{A}(B,A) \to FB \multimap FA$$

*and*

$$(TF)_{AB} : \mathbf{A}(B,A) \to TFB \multimap TFA.$$

*We have the following commutative diagram.*

$$\begin{tikzcd}
\mathbf{A}(B,A) \otimes TFA \arrow[d, "t"] \arrow[dr, "\mathrm{uncurry}((TF)_{AB})"] & \\
T(\mathbf{A}(B,A) \otimes FA) \arrow[r, "T\mathrm{uncurry}(F_{AB})"'] & TFB
\end{tikzcd}$$

*Proof.* By currying the diagram above, we just need to show the right triangle commutes in the following diagram.

$$\begin{tikzcd}
 & \mathbf{A}(B,A) \arrow[ddl, bend right, "F_{AB}"'] \arrow[d, "\eta"] \arrow[ddr, bend left, "(TF)_{AB}"] \arrow[drr, "\mathrm{curry}(t)"] & & \\
 & FA \multimap \mathbf{A}(B,A) \otimes FA \arrow[rr, "T_{FA,\mathbf{A}(B,A)\otimes FA}"] \arrow[dl, "FA \multimap \mathrm{uncurry}(F_{AB})"] & & TFA \multimap T(\mathbf{A}(B,A) \otimes FA) \arrow[dl, "TFA \multimap T\mathrm{uncurry}(F_{AB})"] \\
FA \multimap FB \arrow[rr, "T_{FA,FB}"'] & & TFA \multimap TFB &
\end{tikzcd}$$

Note that the bottom square commutes because of the $\mathscr{V}$-naturality of $T$. The left triangle commutes by the property of monoidal closedness. The front triangle commutes by definition of $(TF)_{AB}$. The back triangle commutes by Proposition D.1. $\square$

**Theorem D.3.** *Let $F : \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathscr{V}$ be a $\mathscr{V}$-functor and let $T$ be a strong monad on $\mathscr{V}$. Then there exists a natural map*

$$\xi_F : \int^{A\in\mathbf{A}} TF(A,A) \to T \int^{A\in\mathbf{A}} F(A,A).$$

*Proof.* Recall that by definition of coend, we have the following coequalizers.

$$\textstyle\sum_{A,B\in\mathbf{A}} \mathbf{A}(B,A)\otimes F(A,B) \underset{\rho_2}{\overset{\rho_1}{\rightrightarrows}} \sum_{A\in\mathbf{A}} F(A,A) \xrightarrow{e} \int^{A\in\mathbf{A}} F(A,A)$$

$$\textstyle\sum_{A,B\in\mathbf{A}} \mathbf{A}(B,A)\otimes TF(A,B) \underset{\rho_2'}{\overset{\rho_1'}{\rightrightarrows}} \sum_{A\in\mathbf{A}} TF(A,A) \xrightarrow{e'} \int^{A\in\mathbf{A}} TF(A,A)$$

For any $A\in\mathbf{A}$, the functor $F(A,-):\mathbf{A}\to\mathscr{V}$ gives rise to a map

$$F(A,-)_{BA}:\mathbf{A}(B,A)\to F(A,B)\multimap F(A,A)$$

for each $B\in\mathbf{A}$. The map $\rho_1$ is defined as the coproduct pairing $[\mathrm{inj}_A\circ\mathrm{uncurry}(F(A,-)_{BA})]_{A,B\in\mathbf{A}}$. For any $B\in\mathbf{A}$, the functor $F(-,B):\mathbf{A}^{\mathrm{op}}\to\mathscr{V}$ gives rise to a map

$$F(-,B)_{AB}:\mathbf{A}(B,A)\to F(A,B)\multimap F(B,B)$$

for each $A\in\mathbf{A}$. The map $\rho_2$ is defined as the coproduct pairing $[\mathrm{inj}_B\circ\mathrm{uncurry}(F(-,B)_{AB})]_{A,B\in\mathbf{A}}$. The maps $\rho_1',\rho_2'$ are induced similarly.

Consider the following diagram.

$$\begin{tikzcd}
 & \int^A TF(A,A) \arrow[rr, dashed, "\xi"] & & T\int^A F(A,A) \\
 & \sum_A TF(A,A) \arrow[u, "e'"] \arrow[rr, "{[T\mathrm{inj}_A]_A}"] & & T\sum_A F(A,A) \arrow[u, "Te"'] \\
\sum_{A,B}\mathbf{A}(B,A)\otimes TF(A,B) \arrow[ur, shift left, "\rho_1'"] \arrow[ur, shift right, "\rho_2'"'] \arrow[r, "\sum_{A,B} t"] & \sum_{A,B} T(\mathbf{A}(B,A)\otimes F(A,B)) \arrow[rr, "{[T\mathrm{inj}_{A,B}]_{A,B}}"] & & T\sum_{A,B}\mathbf{A}(B,A)\otimes F(A,B) \arrow[u, shift left, "T\rho_1"] \arrow[u, shift right, "T\rho_2"']
\end{tikzcd}$$

Note that $[T\mathrm{inj}_A]_A$ and $[T\mathrm{inj}_{A,B}]_{A,B}$ are coproduct pairings. The morphism $t:\mathbf{A}(B,A)\otimes TF(A,B)\to T(\mathbf{A}(B,A)\otimes F(A,B))$ is the strength map for $T$.

To show the existence of $\xi$, we just need to show $Te\circ[T\mathrm{inj}_A]_A\circ\rho_1' = Te\circ[T\mathrm{inj}_A]_A\circ\rho_2'$, which is to show the bottom square commutes for $\rho_1'$ and $T\rho_1$ ($\rho_2'$ and $T\rho_2$). This is the case because of the following commutative diagram. Note that the left triangle commutes by Theorem D.2.

$$\begin{tikzcd}
\mathbf{A}(B,A)\otimes TF(A,B) \arrow[r, "{\rho_1'(A,B)}"] \arrow[d, "t"] & TF(A,A) \arrow[r, "T\mathrm{inj}_A"] & T\sum_A F(A,A) \\
T(\mathbf{A}(B,A)\otimes F(A,B)) \arrow[ur, "{T\rho_1(A,B)}"] \arrow[rr, "T\mathrm{inj}_{A,B}"'] & & T\sum_{A,B}\mathbf{A}(B,A)\otimes F(A,B) \arrow[u, "T\rho_1"']
\end{tikzcd}$$

Note that $\rho_1'(A,B)$ is a component of $\rho_1'$ and $\rho_1(A,B)$ is a component of $\rho_1$. By the universal property of the coequalizer $e'$, there exists a unique arrow

$$\xi:\int^{A\in\mathbf{A}} TF(A,A)\to T\int^{A\in\mathbf{A}} F(A,A). \qquad \square$$

**Proposition D.4.** *Suppose $F : \mathbf{A}^{\mathrm{op}} \to \mathscr{V}$. For all $B, C \in \mathbf{A}$, the following diagram commutes.*

$$\begin{tikzcd}
\mathbf{A}(C,B) \otimes F(B) \arrow[r, "e_B"] \arrow[d, "\mathsf{uncurry}(F_{BC})"'] & \int^B \mathbf{A}(C,B) \otimes F(B) \arrow[dl, "y"] \\
F(C) &
\end{tikzcd}$$

*Note that $y$ is an isomorphism expressing the Yoneda lemma in the language of coends, and $F_{BC} : \mathbf{A}(C,B) \to F(B) \multimap F(C)$, and $e_B$ is the unit of the coend.*

*Proof sketch.* Note that the map $\mathsf{uncurry}(F_{BC}) : \mathbf{A}(C,B) \otimes F(B) \to F(C)$ is $\mathscr{V}$-natural in $B$. By the universal property of coends, there exists a map $y : \int^B \mathbf{A}(C,B) \otimes F(B) \to F(C)$ such that the diagram above commutes. Moreover, $y$ is an isomorphism [11, Chapter 2.4]. ☐

**Theorem D.5.** *Let $T$ be a strong monad on $\mathscr{V}$. For all $F : \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathscr{V}$, the map $\xi_F : \int^{A \in \mathbf{A}} TF(A,A) \to T \int^{A \in \mathbf{A}} F(A,A)$ makes the following diagrams commute.*

1.

$$\begin{tikzcd}
\int^A F(A,A) \arrow[d, "\eta"] \arrow[dr, "\int \eta"] & \\
T \int^A F(A,A) & \int^A TF(A,A) \arrow[l, "\xi"]
\end{tikzcd}$$

2.

$$\begin{tikzcd}
\int^A TTF(A,A) \arrow[r, "\xi"] \arrow[d, "\int \mu"] & T \int^A TF(A,A) \arrow[r, "T\xi"] & TT \int^A F(A,A) \arrow[d, "\mu"] \\
\int^A TF(A,A) \arrow[rr, "\xi"] & & T \int^A F(A,A)
\end{tikzcd}$$

3. *Suppose $G : \mathbf{A}^{\mathrm{op}} \to \mathscr{V}$ and $A \in \mathbf{A}$.*

$$\begin{tikzcd}
\int^B \mathbf{A}(A,B) \otimes TGB \arrow[r, "y'"] \arrow[d, "\int t"] & TGA \\
\int^B T(\mathbf{A}(A,B) \otimes GB) \arrow[r, "\xi"] & T \int^B \mathbf{A}(A,B) \otimes GB \arrow[u, "Ty"']
\end{tikzcd}$$

*Note that $y', y$ are isomorphisms induced by the Yoneda lemma.*

4. *Suppose $F : \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathscr{V}$ and $X \in \mathscr{V}$.*

$$\begin{tikzcd}
(\int^A F(A,A)) \otimes TX \arrow[r, "t"] \arrow[d, "\cong"] & T((\int^A F(A,A)) \otimes X) \arrow[dd, "\cong"] \\
\int^A (F(A,A) \otimes TX) \arrow[d, "\int t"] & \\
\int^A T(F(A,A) \otimes X) \arrow[r, "\xi"] & T \int^A (F(A,A) \otimes X)
\end{tikzcd}$$

5. *Suppose $F : \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathscr{V}$ and $X \in \mathscr{V}$.*

$$\begin{tikzcd}
\int^A (X \otimes TF(A,A)) \arrow[r, "\int t"] \arrow[d, "\cong"] & \int^A T(X \otimes F(A,A)) \arrow[d, "\xi"] \\
X \otimes \int^A TF(A,A) \arrow[d, "X \otimes \xi"] & T \int^A (X \otimes F(A,A)) \arrow[d, "\cong"] \\
X \otimes T \int^A F(A,A) \arrow[r, "t"] & T(X \otimes \int^A F(A,A))
\end{tikzcd}$$

6. *Suppose $F : \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \otimes \mathbf{A} \to \mathscr{V}$.*

$$\begin{tikzcd}
\int^A \int^B TF(A,B,A,B) \arrow[r, "\int^A \xi"] \arrow[d, "\cong"] & \int^A T \int^B F(A,B,A,B) \arrow[r, "\xi"] & T \int^A \int^B F(A,B,A,B) \arrow[d, "\cong"] \\
\int^{A,B} TF(A,B,A,B) \arrow[rr, "\xi"] & & T \int^{A,B} F(A,B,A,B)
\end{tikzcd}$$

*Proof.* 1. We need to show that the following commutes.

$$\begin{tikzcd}
\int^A F(A,A) \arrow[d, "\eta"] \arrow[dr, "\int \eta"] & \\
T \int^A F(A,A) & \int^A TF(A,A) \arrow[l, "\xi"]
\end{tikzcd}$$

Consider the following diagram. We write $\eta_1$ for the map $F(A,A) \to TF(A,A)$ and $\eta_2$ for the map $\int^A F(A,A) \to T \int^A F(A,A)$.

$$\begin{tikzcd}
\int^A F(A,A) \arrow[r, "\int \eta_1"] \arrow[drr, "\eta_2"'] & \int^A TF(A,A) \arrow[dr, "\xi"] & \\
& & T \int^A F(A,A) \\
\sum_A F(A,A) \arrow[r, "\sum \eta_1"] \arrow[uu, two heads, "e"] \arrow[drr, "\eta"'] & \sum_A TF(A,A) \arrow[uu, "e'"] \arrow[dr, "{[T\mathrm{inj}_A]_A}"] & \\
& & T \sum_A F(A,A) \arrow[uu, "Te"]
\end{tikzcd}$$

We need to show that the top triangle commutes. Since $e$ is an epimorphism, we just need to show $\xi \circ \int \eta_1 \circ e = \eta_2 \circ e$. This is the case because the bottom triangle commutes and all three square faces commute. The bottom triangle commutes by the universal property of coproducts. The square with $\xi$ commutes by definition of $\xi$. Also note that $e' \circ \sum \eta_1 = \int \eta_1 \circ e$ is a property of coends (see [11, 4.2]).

2. The proof is similar to (1).

3. Next we need to show that the following commutes (where $y, y'$ are isomorphisms induced by the Yoneda lemma).

$$\begin{tikzcd}
\int^B \mathbf{A}(A,B)\otimes TGB \arrow[r, "\int t"] \arrow[d, "y'"] & \int^B T(\mathbf{A}(A,B)\otimes GB) \arrow[d, "\xi"] \\
TGA & T\int^B \mathbf{A}(A,B)\otimes GB \arrow[l, "Ty"]
\end{tikzcd}$$

The above diagram commutes because the following diagram commutes for all $A, B \in \mathbf{A}$.

$$\begin{tikzcd}
 & \mathbf{A}(A,B)\otimes TGB \arrow[rr, two heads, "e_1"] \arrow[dl, "\mathrm{uncurry}((TG)_{BA})"'] \arrow[d, "t"] & & \int^B \mathbf{A}(A,B)\otimes TGB \arrow[dl, "y'"'] \arrow[d, "\int t"] \\
TGA \arrow[rr, dashed, bend right, "\mathrm{Id}"] & T(\mathbf{A}(A,B)\otimes GB) \arrow[rr, "e_2"] \arrow[dl, "\mathrm{Id}"] & TGA & \int^B T(\mathbf{A}(A,B)\otimes GB) \arrow[dl, "\xi"] \\
T(\mathbf{A}(A,B)\otimes G(B)) \arrow[u, "T\mathrm{uncurry}(G_{BA})"] \arrow[rr, "Te_3"] & & T\int^B \mathbf{A}(A,B)\otimes GB \arrow[u, "Ty"] &
\end{tikzcd}$$

Since $G$ and $TG$ are contravariant $\mathscr{V}$-functors, there are the following maps in $\mathscr{V}$.

$$G_{BA} : \mathbf{A}(A,B) \to GB \Rightarrow GA$$

$$(TG)_{BA} : \mathbf{A}(A,B) \to TGB \Rightarrow TGA$$

The bottom square commutes by the definition of $\xi$, and the back square (with $e_1, e_2$) commutes by naturality of coends. The top and the front squares commutes because of Proposition D.4. Thus we just need to show that the left square commutes, i.e.,

$$\begin{tikzcd}
\mathbf{C}(C,B)\otimes TFB \arrow[r, "t"] \arrow[d, "u'"] & T(\mathbf{C}(C,B)\otimes FB) \arrow[dl, "Tu"] \\
TFC &
\end{tikzcd}.$$

This commutes by Proposition D.2.

4. Next we need to prove that the following commutes.

$$\begin{tikzcd}
(\int^A F(A,A))\otimes TX \arrow[r, "t"] \arrow[d, "\cong"] & T((\int^A F(A,A))\otimes X) \arrow[dd, "\cong"] \\
\int^A (F(A,A)\otimes TX) \arrow[d, "\int t"] & \\
\int^A T(F(A,A)\otimes X) \arrow[r, "\xi"] & T\int^A (F(A,A)\otimes X)
\end{tikzcd}$$

First observe that the following commutes (each arrow is canonical).

$$\begin{tikzcd}
(\sum_A F(A,A))\otimes TX \arrow[r] \arrow[d, "\cong"] & T((\sum_A F(A,A))\otimes X) \arrow[dd, "\cong"] \\
\sum_A (F(A,A)\otimes TX) \arrow[d] & \\
\sum_A T(F(A,A)\otimes X) \arrow[r] & T\sum_A (F(A,A)\otimes X)
\end{tikzcd}$$

Now let us consider the following cube.

$$\begin{tikzcd}
 & (\int^A F(A,A))\otimes TX \arrow[rr, "t"] \arrow[dd, "\cong"] & & T((\int^A F(A,A))\otimes X) \arrow[dddd, "\cong"] \\
(\sum_A F(A,A))\otimes TX \arrow[ur, two heads] \arrow[rr, "t"] \arrow[dd, "\cong"] & & T((\sum_A F(A,A))\otimes X) \arrow[ur, "T(e\otimes X)"] \arrow[dddd, "\cong"] & \\
 & \int^A (F(A,A)\otimes TX) \arrow[dd, "\int t"] & & \\
\sum_A (F(A,A)\otimes TX) \arrow[ur, two heads] \arrow[dd, "\sum_A t"] & & & \\
 & \int^A T(F(A,A)\otimes X) \arrow[rr, "\xi"] & & T\int^A (F(A,A)\otimes X) \\
\sum_A T(F(A,A)\otimes X) \arrow[ur, two heads] \arrow[rr, "{[T\mathrm{inj}_A]_A}"] & & T\sum_A (F(A,A)\otimes X) \arrow[ur, "Te'"] &
\end{tikzcd}$$

Note that the top square commutes, by naturality of $t$. The bottom square commutes, by definition of $\xi$. The left square involving $\sum_A t, \int t$ commutes by naturality of coend. The right square and the left top square commute for the same reason. For simplicity, consider the following diagram.

$$\begin{tikzcd}
(\sum_{A,B}\mathbf{A}(B,A)\otimes F(A,B))\otimes X \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & (\sum_A F(A,A))\otimes X \arrow[r, two heads] \arrow[d, "\cong"] & (\int^A F(A,A))\otimes X \arrow[d, "\cong"] \\
\sum_{A,B}(\mathbf{A}(B,A)\otimes F(A,B)\otimes X) \arrow[r, shift left] \arrow[r, shift right] & \sum_A (F(A,A)\otimes X) \arrow[r, two heads] & \int^A (F(A,A)\otimes X)
\end{tikzcd}$$

Note that the right square is the same square as the right square in the cube. And $-\otimes X$ preserves coequalizers. The left square commutes by naturality. This implies that the right square commutes, by the universal property of coequalizers. Therefore the cube above commutes.

5. Next, we need to show that the following diagram commutes.

$$\begin{tikzcd}
\int^A (X\otimes TF(A,A)) \arrow[r, "\int t"] \arrow[d, "\cong"] & \int^A T(X\otimes F(A,A)) \arrow[d, "\xi"] \\
X\otimes \int^A TF(A,A) \arrow[d, "\mathrm{Id}\otimes\xi"] & T\int^A (X\otimes F(A,A)) \arrow[d, "\cong"] \\
X\otimes T\int^A F(A,A) \arrow[r, "t"] & T(X\otimes \int^A F(A,A))
\end{tikzcd}$$

First, observe that the following diagram commutes.

$$\begin{tikzcd}
\sum_A (X\otimes TF(A,A)) \arrow[r] \arrow[d, "\cong"] & \sum_A T(X\otimes F(A,A)) \arrow[d] \\
X\otimes \sum_A TF(A,A) \arrow[d] & T\sum_A (X\otimes F(A,A)) \arrow[d, "\cong"] \\
X\otimes T\sum_A F(A,A) \arrow[r] & T(X\otimes \sum_A F(A,A))
\end{tikzcd}$$

Now let us consider the following cube.

$$\begin{tikzcd}
& \int^A (X \otimes TF(A,A)) \arrow[rr, "\int t"] \arrow[dd, "\cong"] & & \int^A T(X \otimes F(A,A)) \arrow[dd, "\xi"] \\
\sum_A (X \otimes TF(A,A)) \arrow[ur, two heads] \arrow[rr, "\sum t"] \arrow[dd, "\cong"] & & \sum_A T(X \otimes F(A,A)) \arrow[ur] \arrow[dd] & \\
& X \otimes \int^A TF(A,A) \arrow[dd, "X\otimes\xi"] & & T\int^A (X \otimes F(A,A)) \arrow[dd, "\cong"] \\
X \otimes \sum_A TF(A,A) \arrow[ur] \arrow[dd] & & T\sum_A (X \otimes F(A,A)) \arrow[ur] \arrow[dd, "\cong"] & \\
& X \otimes T\int^A F(A,A) \arrow[rr, "t"] & & T(X \otimes \int^A (F(A,A) \\
X \otimes T\sum_A F(A,A) \arrow[ur, two heads] \arrow[rr, "t"] & & T(X \otimes \sum_A F(A,A)) \arrow[ur, "Te'"] &
\end{tikzcd}$$

Note that the top square commutes by naturality of coends. The bottom square commutes by naturality of $t$. The left bottom and the right top square involving $\xi$ commute by definition. The left top and the right bottom square commute for the same reason. Consider the following diagram.

$$\begin{tikzcd}
\sum_{A,B} (\mathbf{A}(B,A) \otimes X \otimes F(A,B)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \sum_A (X \otimes F(A,A)) \arrow[r, two heads] \arrow[d, "\cong"] & \int^A (X \otimes F(A,A)) \arrow[d, "\cong"] \\
X \otimes \sum_{A,B} (\mathbf{A}(B,A) \otimes F(A,B)) \arrow[r, shift left] \arrow[r, shift right] & X \otimes \sum_A F(A,A) \arrow[r, two heads] & X \otimes \int^A F(A,A)
\end{tikzcd}$$

Note that the right square is the same square as the right bottom square in the cube under the functor $T$. And $X \otimes -$ preserves coequalizers. The left square commutes by naturality. This implies that the right square commutes, by the universal property of coequalizers. Therefore the cube above commutes.

6. Let $F : \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \otimes \mathbf{A} \to \mathscr{V}$. We now need to show that the following diagram commutes.

$$\begin{tikzcd}
\int^A \int^B TF(A,B,A,B) \arrow[r, "\int^A \xi"] \arrow[d, "f"] & \int^A T \int^B F(A,B,A,B) \arrow[r, "\xi"] & T\int^A \int^B F(A,B,A,B) \arrow[d, "Tf"] \\
\int^{A,B} TF(A,B,A,B) \arrow[rr, "\xi"] & & T\int^{A,B} F(A,B,A,B)
\end{tikzcd}$$

First, the isomorphisms $f$, $Tf$ above are instances of so-called Fubini theorem for coends, which also gives rise to the following commutative diagram for any $F : \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \otimes \mathbf{A} \to \mathscr{V}$.

$$\begin{tikzcd}
F(A,B,A,B) \arrow[r, two heads, "e_{A,B}"] \arrow[d, two heads, "e_B"] & \int^{A,B} F(A,B,A,B) \\
\int^B F(A,B,A,B) \arrow[r, two heads, "e_A"] & \int^A \int^B F(A,B,A,B) \arrow[u, "f"]
\end{tikzcd}$$

We have the following commutative diagram.

$$\begin{tikzcd}
& \int^A\int^B TF(A,B,A,B) \arrow[r, "\int^A \xi"] \arrow[dl, "f"'] & \int^A T\int^B F(A,B,A,B) \arrow[r, "\xi"] & T\int^A\int^B F(A,B,A,B) \arrow[d, "Tf"] \\
\int^{A,B} TF(A,B,A,B) \arrow[rrr, bend right=40, "\xi"] & \int^B TF(A,B,A,B) \arrow[r, "\xi"] \arrow[u, "e'_A"] & T\int^B F(A,B,A,B) \arrow[u, "e''_A"] \arrow[ur, "Te_A"] & T\int^{A,B} F(A,B,A,B) \\
TF(A,B,A,B) \arrow[u, "e'_{A,B}"] \arrow[ur, "e'_B"] \arrow[urr, "Te_B"] \arrow[urrr, "Te_{A,B}"'] & & &
\end{tikzcd}$$

Note that above diagram commutes, by properties of the Fubini theorem, definition of $\xi$, $\mathscr{V}$-naturality of coends, and because $e'_{A,B}$ is an epimorphism.

$\square$

**Theorem D.6.** *Let* $\mathbf{A}$ *be a* $\mathscr{V}$*-category. If* $T$ *is a commutative strong monad on* $\mathscr{V}$ *(the strength is given by the map* $t_{A,B} : A \otimes TB \to T(A \otimes B)$ *for any* $A,B \in \mathbf{A}$*), then* $\overline{T}(F) = T \circ F$ *is a commutative strong* $\mathscr{V}$*-monad on* $\mathscr{V}^{\mathbf{A}^{\mathrm{op}}}$.

*Proof.* It is straightforward to verify that $\overline{T}$ is a monad. We define the strength $\bar{t}$ to be the following composition.

$$\begin{aligned}
(F \otimes_{\mathrm{Day}} \overline{T}G)(C) &= \int^{(A,B)\in\mathbf{A}\otimes\mathbf{A}} \mathbf{A}(C, A\otimes B) \otimes FA \otimes TGB \\
&\xrightarrow{\int^{(A,B)} t} \int^{(A,B)\in\mathbf{A}\otimes\mathbf{A}} T(\mathbf{A}(C, A\otimes B) \otimes FA \otimes GB) \\
&\xrightarrow{\xi} T\int^{(A,B)\in\mathbf{A}\otimes\mathbf{A}} (\mathbf{A}(C, A\otimes B) \otimes FA \otimes GB) \\
&= \overline{T}(F \otimes_{\mathrm{Day}} G)(C)
\end{aligned}$$

Now to show that $\overline{T}$ is a commutative strong monad, we need to show the following diagrams commute.

- $$\begin{tikzcd}
F \otimes_{\mathrm{Day}} G \arrow[r, "F\otimes_{\mathrm{Day}}\eta"] \arrow[d, "\eta"] & F \otimes_{\mathrm{Day}} \overline{T}G \arrow[dl, "\bar{t}"] \\
\overline{T}(F \otimes_{\mathrm{Day}} G) &
\end{tikzcd}$$

  To show this, we just need to show that the following diagram commutes for any $C \in \mathbf{A}$.

$$\begin{tikzcd}
\int^{A,B} \mathbf{A}(C,A\otimes B)\otimes FA\otimes GB \arrow[r, "\int^{A,B}\eta'"] \arrow[d, "\eta"] \arrow[dr, "\int^{A,B}\eta"] & \int^{A,B} \mathbf{A}(C,A\otimes B)\otimes FA\otimes TGB \arrow[d, "\int^{A,B} t"] \\
T(\int^{A,B} \mathbf{A}(C,A\otimes B)\otimes FA\otimes GB) & \int^{A,B} T(\mathbf{A}(C,A\otimes B)\otimes FA\otimes GB) \arrow[l, "\xi"]
\end{tikzcd}$$

  Note that $\int^{A,B}\eta'$ is a shorthand for $\int^{A,B}\mathbf{A}(C,A\otimes B)\otimes FA\otimes\eta$. Similarly, $\int^{A,B} t$ is a shorthand for $\int^{A,B} t_{\mathbf{A}(C,A\otimes B)\otimes FA, GB}$ in the above diagram. The bottom triangle commutes because of Theorem D.5(1). The top triangle commutes by properties of $t$.

- We need to show that the following diagram commutes.

$$\begin{tikzcd}
I \otimes_{\mathrm{Day}} \overline{T}F \arrow[r, "\bar{t}_{I,F}"] \arrow[dr, "\lambda_{\overline{T}F}"'] & \overline{T}(I \otimes_{\mathrm{Day}} F) \arrow[d, "\overline{T}\lambda_F"] \\
& \overline{T}F
\end{tikzcd}$$

If we unfold the definition of the Day tensor, we have the following diagram for any $C \in \mathbf{A}$.

$$\begin{tikzcd}
& \int^B \mathbf{A}(C, I \otimes B) \otimes TFB \arrow[r, "\int t"] \arrow[dl, "\int \lambda"'] & \int^B T(\mathbf{A}(C, I \otimes B) \otimes FB) \arrow[dl, "\int T\lambda"'] \arrow[dr, "\xi"] & \\
\int^B \mathbf{A}(C,B) \otimes TFB \arrow[r, "\int t'"] \arrow[dr, "\cong"] & \int^B T(\mathbf{A}(C,B) \otimes FB) \arrow[dr, "\xi"] & & T \int^B \mathbf{A}(C, I \otimes B) \otimes FB \arrow[dl, "T\int\lambda"] \\
& TFC \arrow[r, "\cong"] & T \int^B \mathbf{A}(C,B) \otimes FB &
\end{tikzcd}$$

Note that for any $C \in \mathbf{A}$, the following commutes by naturality of $t$.

$$\begin{tikzcd}
\int^B \mathbf{A}(C, I \otimes B) \otimes TFB \arrow[r, "\int t"] \arrow[d, "\int \lambda_B"'] & \int^B T(\mathbf{A}(C, I \otimes B) \otimes FB) \arrow[d, "\int T\lambda_B"] \\
\int^B \mathbf{A}(C,B) \otimes TFB \arrow[r, "\int t'"] & \int^B T(\mathbf{A}(C,B) \otimes FB)
\end{tikzcd}$$

Note that $I = \mathbf{A}(-,I) \in \mathscr{V}^{\mathbf{A}^{\mathrm{op}}}$, and $\int t$ is a shorthand for $\int^B t_{\mathbf{A}(C,I\otimes B),FB}$, and $\int \lambda_B$ is a shorthand for $\int^B \mathbf{A}(C,\lambda_B) \otimes \mathrm{Id}_{TFB}$, and $\int T\lambda_B$ is a shorthand for $\int^B T\mathbf{A}(C,\lambda_B) \otimes \mathrm{Id}_{FB}$, and $\int t'$ is a shorthand for $\int^B t_{\mathbf{A}(C,B),FB}$.
The bottom square commutes because of Theorem D.5(3). The right square commutes by the naturality of $\xi$.

- Next we need to show that the following diagram commutes.

$$\begin{tikzcd}
F \otimes_{\mathrm{Day}} \overline{TT}G \arrow[r, "t"] \arrow[d, "\mathrm{Id}_F \otimes_{\mathrm{Day}} \mu"'] & \overline{T}(F \otimes_{\mathrm{Day}} \overline{T}G) \arrow[r, "\overline{T}t"] & \overline{TT}(F \otimes_{\mathrm{Day}} G) \arrow[d, "\mu"] \\
F \otimes_{\mathrm{Day}} \overline{T}G \arrow[rr, "t"] & & \overline{T}(F \otimes_{\mathrm{Day}} G)
\end{tikzcd}$$

The above diagram commutes because for any $C \in \mathbf{A}$, we have the following commutative diagram.

$$\begin{tikzcd}
\int^{A,B} \mathbf{A}(C, A \otimes B) \otimes FA \otimes TTGB \arrow[r, "\int t"] \arrow[d, "\int \mathrm{Id} \otimes \mu"'] & \int^{A,B} T(\mathbf{A}(C, A \otimes B) \otimes FA \otimes TGB) \arrow[r, "\xi"] \arrow[d, "\int Tt"] & T \int^{A,B} (\mathbf{A}(C, A \otimes B) \otimes FA \otimes TGB) \arrow[d, "T\int t"] \\
\int^{A,B} \mathbf{A}(C, A \otimes B) \otimes FA \otimes TGB \arrow[d, "\int t"'] & \int^{A,B} TT(\mathbf{A}(C, A \otimes B) \otimes FA \otimes GB) \arrow[r, "\xi"] \arrow[dl, "\int \mu"'] & T \int^{A,B} T(\mathbf{A}(C, A \otimes B) \otimes FA \otimes GB) \arrow[d, "T\xi"] \\
\int^{A,B} T(\mathbf{A}(C, A \otimes B) \otimes FA \otimes GB) \arrow[r, "\xi"] & T \int^{A,B} (\mathbf{A}(C, A \otimes B) \otimes FA \otimes GB) & TT \int^{A,B} (\mathbf{A}(C, A \otimes B) \otimes FA \otimes GB) \arrow[l, "\mu"]
\end{tikzcd}$$

Note that the top right square commutes by naturality of $\xi$, the bottom diagram commutes by Theorem D.5 (2), and the left diagram commutes by properties of $t$.

- Next we need to show that the following diagram commutes.

$$\begin{tikzcd}
(F \otimes_{\text{Day}} G) \otimes_{\text{Day}} \overline{T}H \arrow[rr, "t"] \arrow[d, "\alpha"] & & \overline{T}((F \otimes_{\text{Day}} G) \otimes_{\text{Day}} H) \arrow[d, "\overline{T}\alpha"] \\
F \otimes_{\text{Day}} (G \otimes_{\text{Day}} \overline{T}H) \arrow[r, "\mathrm{Id}_F \otimes_{\text{Day}} t"] & F \otimes_{\text{Day}} \overline{T}(G \otimes_{\text{Day}} H) \arrow[r, "t"] & \overline{T}(F \otimes_{\text{Day}} (G \otimes_{\text{Day}} H))
\end{tikzcd}$$

For any $C \in \mathbf{A}$, we have

$$((F \otimes_{\text{Day}} G) \otimes_{\text{Day}} \overline{T}H)(C) \cong \int^{B \in \mathbf{A}} \int^{(X,Y) \in \mathbf{A} \otimes \mathbf{A}} \mathbf{A}(C, (X \otimes Y) \otimes B) \otimes FX \otimes GY \otimes THB$$

and

$$(F \otimes_{\text{Day}} (G \otimes_{\text{Day}} \overline{T}H))(C) \cong \int^{X \in \mathbf{A}} \int^{(Y,B) \in \mathbf{A} \otimes \mathbf{A}} \mathbf{A}(C, X \otimes (Y \otimes B)) \otimes FX \otimes GY \otimes THB$$

$$\cong \int^{X,Y,B} FX \otimes GY \otimes THB \otimes \int^{A} \mathbf{A}(A, Y \otimes B) \otimes \mathbf{A}(C, X \otimes A).$$

Consider the following diagram. We need to show that the outermost diagram commutes. Note that $\int y, T\int y, a, a'$ are all isomorphisms.

$$\begin{tikzcd}
\int^{B,X,Y} \mathbf{A}(C,(X\otimes Y)\otimes B)\otimes FX\otimes GY\otimes THB \arrow[rr, "\int t"] \arrow[d, "\int\alpha"] & & \int^{B,X,Y} T(\mathbf{A}(C,(X\otimes Y)\otimes B)\otimes FX\otimes GY\otimes HB) \arrow[dl, "\int T\alpha"'] \arrow[dr, "\xi"] & \\
\int^{B,X,Y} \mathbf{A}(C,X\otimes (Y\otimes B))\otimes FX\otimes GY\otimes THB \arrow[r, "\int t"] \arrow[d, "\cong"] & \int^{B,X,Y} T(\mathbf{A}(C,X\otimes (Y\otimes B))\otimes FX\otimes GY\otimes HB) \arrow[dr, "\xi"] & & T\int^{B,X,Y} \mathbf{A}(C,(X\otimes Y)\otimes B)\otimes FX\otimes GY\otimes HB \arrow[dl, "T\int\alpha"] \\
\int^{A,X} \mathbf{A}(C,X\otimes A)\otimes FX\otimes \int^{Y,B} \mathbf{A}(A,Y\otimes B)\otimes GY\otimes THB \arrow[d, "\int \mathrm{Id}\otimes\int t"] & & T\int^{B,X,Y} \mathbf{A}(C,X\otimes (Y\otimes B))\otimes FX\otimes GY\otimes HB \arrow[d, "\cong"] & \\
\int^{A,X} \mathbf{A}(C,X\otimes A)\otimes FX\otimes \int^{Y,B} T(\mathbf{A}(A,Y\otimes B)\otimes GY\otimes HB) \arrow[d, "\int \mathrm{Id}\otimes\xi"] & & T\int^{X,Y,B}\int^{A} \mathbf{A}(C,X\otimes A)\otimes \mathbf{A}(A,Y\otimes B)\otimes FX\otimes GY\otimes HB \arrow[d, "\cong"] & \\
\int^{A,X} \mathbf{A}(C,X\otimes A)\otimes FX\otimes T\int^{Y,B} \mathbf{A}(A,Y\otimes B)\otimes GY\otimes HB \arrow[r, "\int t"] & \int^{A,X} T(\mathbf{A}(C,X\otimes A)\otimes FX\otimes \int^{Y,B} \mathbf{A}(A,Y\otimes B)\otimes GY\otimes HB) \arrow[r, "\xi"] & T\int^{A,X} \mathbf{A}(C,X\otimes A)\otimes FX\otimes \int^{Y,B} \mathbf{A}(A,Y\otimes B)\otimes GY\otimes HB &
\end{tikzcd}$$

Note that the top square and the top right square commute by naturality of $t$ and $\xi$. We just need to show that the bottom diagram commutes. The expanded bottom diagram is the following.

$$\begin{tikzcd}
\int^{B,X,Y} \mathbf{A}(C,X\otimes (Y\otimes B))\otimes FX\otimes GY\otimes THB \arrow[rr, "\int t"] \arrow[d, "\cong"] & & \int^{B,X,Y} T(\mathbf{A}(C,X\otimes (Y\otimes B))\otimes FX\otimes GY\otimes HB) \arrow[r, "\xi"] \arrow[d, "\cong"] & T\int^{B,X,Y} \mathbf{A}(C,X\otimes (Y\otimes B))\otimes FX\otimes GY\otimes HB \arrow[d, "\cong"] \\
\int^{B,X,Y} (\int^{A} \mathbf{A}(A,Y\otimes B)\otimes \mathbf{A}(C,X\otimes A))\otimes FX\otimes GY\otimes THB \arrow[rr, "\int t"] \arrow[d, "\cong"] & & \int^{B,X,Y} T((\int^{A} \mathbf{A}(A,Y\otimes B)\otimes \mathbf{A}(C,X\otimes A))\otimes FX\otimes GY\otimes HB) \arrow[r, "\xi"] \arrow[d, "\cong"] & T\int^{B,X,Y} (\int^{A} \mathbf{A}(A,Y\otimes B)\otimes \mathbf{A}(C,X\otimes A))\otimes FX\otimes GY\otimes HB \arrow[d, "\cong"] \\
\int^{B,X,Y}\int^{A} (\mathbf{A}(A,Y\otimes B)\otimes \mathbf{A}(C,X\otimes A)\otimes FX\otimes GY\otimes THB) \arrow[r, "\int\int t"] \arrow[d, "\cong"] & \int^{B,X,Y}\int^{A} T(\mathbf{A}(A,Y\otimes B)\otimes \mathbf{A}(C,X\otimes A)\otimes FX\otimes GY\otimes HB) \arrow[r, "\int\xi"] \arrow[d, "\cong"] & \int^{B,X,Y} T\int^{A} (\mathbf{A}(A,Y\otimes B)\otimes \mathbf{A}(C,X\otimes A)\otimes FX\otimes GY\otimes HB) \arrow[r, "\xi"] & T\int^{B,X,Y}\int^{A} (\mathbf{A}(A,Y\otimes B)\otimes \mathbf{A}(C,X\otimes A)\otimes FX\otimes GY\otimes HB) \arrow[d, "\cong"] \\
\int^{B,Y}\int^{A,X} (\mathbf{A}(A,Y\otimes B)\otimes \mathbf{A}(C,X\otimes A)\otimes FX\otimes GY\otimes THB) \arrow[r, "\int\int t"] \arrow[d, "\cong"] & \int^{B,Y}\int^{A,X} T(\mathbf{A}(A,Y\otimes B)\otimes \mathbf{A}(C,X\otimes A)\otimes FX\otimes GY\otimes HB) \arrow[r, "\int\xi"] \arrow[d, "\cong"] & \int^{B,Y} T\int^{A,X} (\mathbf{A}(A,Y\otimes B)\otimes \mathbf{A}(C,X\otimes A)\otimes FX\otimes GY\otimes HB) \arrow[r, "\xi"] & T\int^{B,Y}\int^{A,X} (\mathbf{A}(A,Y\otimes B)\otimes \mathbf{A}(C,X\otimes A)\otimes FX\otimes GY\otimes HB) \arrow[d, "\cong"] \\
\int^{A,X}\int^{B,Y} (\mathbf{A}(C,X\otimes A)\otimes FX\otimes \mathbf{A}(A,Y\otimes B)\otimes GY\otimes THB) \arrow[r, "\int\int t"] \arrow[d, "\cong"] \arrow[dr, "\int\int \mathrm{Id}\otimes t"'] & \int^{A,X}\int^{B,Y} T(\mathbf{A}(C,X\otimes A)\otimes FX\otimes \mathbf{A}(A,Y\otimes B)\otimes GY\otimes HB) \arrow[r, "\int\xi"] & \int^{A,X} T\int^{B,Y} (\mathbf{A}(C,X\otimes A)\otimes FX\otimes \mathbf{A}(A,Y\otimes B)\otimes GY\otimes HB) \arrow[r, "\xi"] \arrow[dd, "\cong"] & T\int^{A,X}\int^{B,Y} (\mathbf{A}(C,X\otimes A)\otimes FX\otimes \mathbf{A}(A,Y\otimes B)\otimes GY\otimes HB) \arrow[dd, "\cong"] \\
\int^{A,X} \mathbf{A}(C,X\otimes A)\otimes FX\otimes \int^{B,Y} (\mathbf{A}(A,Y\otimes B)\otimes GY\otimes THB) \arrow[d, "\int \mathrm{Id}\otimes\int t"] & \int^{A,X}\int^{B,Y} (\mathbf{A}(C,X\otimes A)\otimes FX\otimes T(\mathbf{A}(A,Y\otimes B)\otimes GY\otimes HB)) \arrow[u, "\int\int t"] \arrow[dl, "\cong"] & & \\
\int^{A,X} \mathbf{A}(C,X\otimes A)\otimes FX\otimes \int^{B,Y} T(\mathbf{A}(A,Y\otimes B)\otimes GY\otimes HB) \arrow[r, "\int \mathrm{Id}\otimes\xi"] & \int^{A,X} \mathbf{A}(C,X\otimes A)\otimes FX\otimes T\int^{B,Y} (\mathbf{A}(A,Y\otimes B)\otimes GY\otimes HB) \arrow[r, "\int t"] & \int^{A,X} T(\mathbf{A}(C,X\otimes A)\otimes FX\otimes \int^{B,Y} (\mathbf{A}(A,Y\otimes B)\otimes GY\otimes HB)) \arrow[r, "\xi"] & T\int^{A,X} (\mathbf{A}(C,X\otimes A)\otimes FX\otimes \int^{B,Y} (\mathbf{A}(A,Y\otimes B)\otimes GY\otimes HB))
\end{tikzcd}$$

Our goal is to show that the outermost diagram commutes. Note that all the inner diagrams commute, by Theorem D.3(4)–(6) and naturality. Therefore the whole diagram commutes.

- Lastly, since $\mathscr{V}^{\mathbf{A}^{\text{op}}}$ is a symmetric monoidal $\mathscr{V}$-category with $\gamma_{F,G} : F \otimes_{\text{Day}} G \to G \otimes_{\text{Day}} F$, we can define the costrength as the following for any $F, B \in \mathscr{V}^{\mathbf{A}^{\text{op}}}$.

$$\sigma_{F,G} := \overline{T}\gamma_{G,F} \circ t_{G,F} \circ \gamma_{\overline{T}F,G} : \overline{T}F \otimes_{\text{Day}} G \to \overline{T}(F \otimes_{\text{Day}} G)$$

We need to show that the following diagram commutes.

$$\begin{tikzcd}
& \overline{T}F \otimes_{\mathrm{Day}} \overline{T}G \arrow[dl, "\sigma"'] \arrow[dr, "t"] & \\
\overline{T}(F \otimes_{\mathrm{Day}} \overline{T}G) \arrow[d, "\overline{T}t"] & & \overline{T}(\overline{T}F \otimes_{\mathrm{Day}} G) \arrow[d, "\overline{T}\sigma"] \\
\overline{T}\,\overline{T}(F \otimes_{\mathrm{Day}} G) \arrow[dr, "\mu"] & & \overline{T}\,\overline{T}(F \otimes_{\mathrm{Day}} G) \arrow[dl, "\mu"'] \\
& \overline{T}(F \otimes_{\mathrm{Day}} G) &
\end{tikzcd}$$

For any $C \in \mathbf{A}$, the above diagram can be expanded to the following diagram. We need to show the outermost diagram commutes.

$$\begin{tikzcd}[column sep=tiny]
& & & & \int^{A,B}\mathbf{A}(C,A\otimes B)\otimes TFA\otimes TGB \arrow[dll, "\int\gamma"'] \arrow[drr, "\int t"] & & & \\
& & \int^{A,B}\mathbf{A}(C,B\otimes A)\otimes TGB\otimes TFA \arrow[dl, "\int t"] & & & & \int^{A,B}T(\mathbf{A}(C,A\otimes B)\otimes TFA\otimes GB) \arrow[dl, "\int T\gamma"] \arrow[dr, "\xi"] & \\
& \int^{A,B}T(\mathbf{A}(C,B\otimes A)\otimes TGB\otimes FA) \arrow[dl, "\xi"] \arrow[dr, "\int T\gamma"] & & & & \int^{A,B}T(\mathbf{A}(C,B\otimes A)\otimes GB\otimes TFA) \arrow[dl, "\int Tt"] \arrow[dr, "\xi"] & & T\int^{A,B}(\mathbf{A}(C,A\otimes B)\otimes TFA\otimes GB) \arrow[dl, "T\int\gamma"] \\
T\int^{A,B}(\mathbf{A}(C,B\otimes A)\otimes TGB\otimes FA) \arrow[dr, "T\int\gamma"] & & \int^{A,B}T(\mathbf{A}(C,A\otimes B)\otimes FA\otimes TGB) \arrow[dl, "\xi"] \arrow[dr, "\int Tt"] & & \int^{A,B}TT(\mathbf{A}(C,B\otimes A)\otimes GB\otimes FA) \arrow[dl, "\int TT\gamma"] \arrow[dr, "\xi"] & & T\int^{A,B}(\mathbf{A}(C,B\otimes A)\otimes GB\otimes TFA) \arrow[dl, "T\int t"] & \\
& T\int^{A,B}(\mathbf{A}(C,A\otimes B)\otimes FA\otimes TGB) \arrow[dr, "T\int t"] & & \int^{A,B}TT(\mathbf{A}(C,A\otimes B)\otimes FA\otimes GB) \arrow[dl, "\xi"] \arrow[ddr, "\int\mu"] & & T\int^{A,B}T(\mathbf{A}(C,B\otimes A)\otimes GB\otimes FA) \arrow[dl, "T\xi"] & & \\
& & T\int^{A,B}T(\mathbf{A}(C,A\otimes B)\otimes FA\otimes GB) \arrow[dr, "T\xi"] & & TT\int^{A,B}(\mathbf{A}(C,B\otimes A)\otimes GB\otimes FA) \arrow[dl, "TT\int\gamma"] & & & \\
& & & TT\int^{A,B}(\mathbf{A}(C,A\otimes B)\otimes FA\otimes GB) \arrow[d, "\mu"] & \int^{A,B}T(\mathbf{A}(C,A\otimes B)\otimes FA\otimes GB) \arrow[dl, "\xi"] & & & \\
& & & T\int^{A,B}(\mathbf{A}(C,A\otimes B)\otimes FA\otimes GB) & & & &
\end{tikzcd}$$

It commutes because every inner diagram commutes (by naturality and Theorem D.3(2)). □

# E Proof of Theorem 4.4

We write $[\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$ for the full subcategory of $\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}$ consisting of product-preserving functors. We write $L : \mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}} \to [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$ for the left adjoint of the inclusion functor $i : [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}} \hookrightarrow \mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}$. We write $\eta : \mathrm{Id} \to iL$ for the unit of the adjunction.

**Definition E.1.** We define a function $\tilde{L} : \overline{\mathbf{C}} \to \widetilde{\mathbf{C}}$ as follows.

- For any $F \in \overline{\mathbf{C}}$, we define $\tilde{L}(F) = G$ such that for all $A \in \mathbf{C}$,

$$G(A) = ((iLF^0)A, (FA)_1, (\eta_{F^0})_A \circ h_A : (FA)_1 \to (iLF^0)A) \in \mathscr{V},$$

  where $h_A : (FA)_1 \to (FA)_0$ and $(\eta_{F^0})_A : F^0A \to (iLF^0)A$.
  For any $A, B \in \mathbf{C}$, we define $G^0_{AB}$ and $G^1_{AB}$ by the following.

$$\begin{tikzcd}
f \in \mathbf{M}^{\mathrm{op}}(A,B) \arrow[r, mapsto, "G^1_{AB}"] \arrow[d, mapsto, "J_{AB}"] & (F^1_{AB}f, (iLF^0)(J_{AB}f)) \in \mathscr{V}(GA, GB) \arrow[d, mapsto, "p_0"] \\
J_{AB}f \in \mathbf{Q}^{\mathrm{op}}(A,B) \arrow[r, mapsto, "G^0_{AB}"] & (iLF^0)(J_{AB}f) \in \mathbf{Set}((iLF^0)A, (iLF^0)B)
\end{tikzcd}$$

  Note that $G$ is smooth, hence $G \in \widetilde{\mathbf{C}}$.

**Proposition E.2.** *$\tilde{L} : \overline{\mathbf{C}} \to \widetilde{\mathbf{C}}$ is a $\mathscr{V}$-functor.*

*Proof.* We just need to show that for all $F, G \in \overline{\mathbf{C}}$, there is a morphism

$$\tilde{L}_{FG} : \overline{\mathbf{C}}(F,G) \to \widetilde{\mathbf{C}}(\tilde{L}F, \tilde{L}G)$$

in $\mathscr{V}$. This is provided by the following commuting square in **Set**.

$$\begin{tikzcd}
\{(\alpha_A : FA \to GA)_{A\in\mathbf{C}} \mid \alpha \in \mathscr{V}\text{-Nat}(F,G)\} \arrow[r, "\tilde{L}^1_{FG}"] \arrow[d] & \{(\beta_A : (i\tilde{L}F)A \to (i\tilde{L}G)A)_{A\in\mathbf{C}} \mid \beta \in \mathscr{V}\text{-Nat}(i\tilde{L}F, i\tilde{L}G)\} \arrow[d] \\
\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}(F^0, G^0) \arrow[r, "\tilde{L}^0_{FG}"] & \left[\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}\right]_{\mathrm{prod}}(LF^0, LG^0)
\end{tikzcd}$$

We write $\mathscr{V}$-Nat$(F,G)$ for the set of $\mathscr{V}$-natural transformations from $F$ to $G$. The arrow $\tilde{L}^0_{FG}$ is given by the functor $L : \mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}} \to [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$. And the arrow $\tilde{L}^1_{FG}$ is given by extending the commuting square $\alpha_A : FA \to GA$ with $\eta : \mathrm{Id} \to iL$, as in the following diagram. Note that for each $\mathscr{V}$-natural transformation $\alpha \in \mathscr{V}\mathrm{Nat}(F,G)$, we have $\alpha^0 : F^0 \to G^0$.

$$\begin{tikzcd}
(FA)_1 \arrow[r, "\alpha^1_A"] \arrow[d] & (GA)_1 \arrow[d] \\
(FA)_0 \arrow[r, "\alpha^0_A"] \arrow[d, "(\eta_{F^0})_A"] & (GA)_0 \arrow[d, "(\eta_{G^0})_A"] \\
(iLF^0)A \arrow[r, "(iL)\alpha^0_A"] & (iLG^0)A
\end{tikzcd}$$

□

**Theorem E.3.** *The $\mathscr{V}$-category $\widetilde{\mathbf{C}}$ is a reflective $\mathscr{V}$-subcategory of $\overline{\mathbf{C}}$, i.e., the inclusion $\mathscr{V}$-functor $i : \widetilde{\mathbf{C}} \hookrightarrow \overline{\mathbf{C}}$ has a left adjoint $\tilde{L}$.*

*Proof sketch.* We need to show $\overline{\mathbf{C}}(F, iG) \cong \widetilde{\mathbf{C}}(\tilde{L}F, G)$ for any $F \in \overline{\mathbf{C}}, G \in \widetilde{\mathbf{C}}$ and it is $\mathscr{V}$-natural in $F$ and $G$. We just need to show the following diagram commutes.

$$\begin{tikzcd}
\mathscr{V}\text{-Nat}(F, iG) \arrow[r, "\cong"] \arrow[d] & \mathscr{V}\text{-Nat}(i\tilde{L}F, iG) \arrow[d] \\
\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}(F^0, iG^0) \arrow[r, "\cong"] & [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}(LF^0, G^0)
\end{tikzcd}$$

The bottom arrow is an isomorphism because $L \dashv i$. The top arrow is an isomorphism because for any $A \in \mathbf{C}$ and $\mathscr{V}$-natural transformation $\gamma : F \to iG$, we have the following commutative diagram.

$$\begin{tikzcd}
(FA)_1 \arrow[r, "\gamma^1_A"] \arrow[d] & ((iG)A)_1 \arrow[d] \\
(FA)_0 \arrow[r, "\gamma^0_A"] \arrow[d, "(\eta_{F^0})_A"'] & ((iG)A)_0 \\
(iLF^0)A \arrow[ur, "i(\widehat{\gamma^0})_A"'] &
\end{tikzcd}$$

□

# F Day's reflection theorem for $\widetilde{\mathbf{C}}$

**Theorem F.1.** *If $H \in \widetilde{\mathbf{C}}$, then $G \multimap_{\mathrm{Day}} iH$ is also a smooth functor for any $G \in \overline{\mathbf{C}}$.*

*Proof.* Suppose $H \in \widetilde{\mathbf{C}}$ and $G \in \overline{\mathbf{C}}$. For any $C \in \mathbf{C}$, we have

$$(G \multimap_{\mathrm{Day}} iH)(C) = \int_{A \in \mathbf{C}} GA \Rightarrow iH(C \otimes A) \cong \overline{\mathbf{C}}(G, iH(C \otimes -)).$$

Thus

$$\begin{aligned}
(G \multimap_{\mathrm{Day}} H)(C)_0 &\cong \overline{\mathbf{C}}(G, iH(C \otimes -))_0 \cong \mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}(G^0, (iH)^0(C \otimes -)) \\
&\cong \int_{A \in \mathbf{Q}} \mathbf{Set}(G^0 A, (iH)^0(C \otimes A)) \cong (G^0 \multimap_{\mathrm{Day}} (iH)^0)(C),
\end{aligned}$$

where $G^0 \multimap_{\mathrm{Day}} (iH)^0$ is an exponential in $\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}$. Since $H^0$ preserves products, so does $G^0 \multimap_{\mathrm{Day}} (iH)^0$, thus $G \multimap_{\mathrm{Day}} iH$ is smooth and $G \multimap_{\mathrm{Day}} iH \in \widetilde{\mathbf{C}}$. The functor $G^0 \multimap_{\mathrm{Day}} (iH)^0$ preserves products in $\mathbf{Q}^{\mathrm{op}}$ because for any $C_1, C_2 \in \mathbf{C}$, we have

$$\begin{aligned}
(G^0 \multimap_{\mathrm{Day}} iH^0)(C_1 + C_2) &= \int_A \mathbf{Set}(G^0 A, iH^0((C_1 + C_2) \otimes A)) \\
&\cong \int_A \mathbf{Set}(G^0 A, iH^0(C_1 \otimes A + C_2 \otimes A)) \\
&\cong \int_A \mathbf{Set}(G^0 A, iH^0(C_1 \otimes A) \times iH^0(C_2 \otimes A)) \\
&\cong \int_A \mathbf{Set}(G^0 A, iH^0(C_1 \otimes A)) \times \mathbf{Set}(G^0 A, iH^0(C_2 \otimes A)) \\
&\cong \int_A \mathbf{Set}(G^0 A, iH^0(C_1 \otimes A)) \times \int_A \mathbf{Set}(G^0 A, iH^0(C_2 \otimes A)) \\
&= (G^0 \multimap_{\mathrm{Day}} iH^0)(C_1) \times (G^0 \multimap_{\mathrm{Day}} iH^0)(C_2).
\end{aligned}$$

□

The above theorem implies that for any $G \in \widetilde{\mathbf{C}}, F \in \overline{\mathbf{C}}$, the unit $\eta_{F \multimap_{\mathrm{Day}} iG} : F \multimap_{\mathrm{Day}} iG \to i\tilde{L}(F \multimap_{\mathrm{Day}} iG)$ is an isomorphism, which gives rise to the following theorem.

**Theorem F.2.** *For any $F, H \in \overline{\mathbf{C}}$, we have*

$$\tilde{L}(F \otimes_{\mathrm{Day}} H) \overset{\tilde{L}(\eta_F \otimes_{\mathrm{Day}} H)}{\cong} \tilde{L}(i\tilde{L}F \otimes_{\mathrm{Day}} H).$$

*Proof.* For any $G \in \widetilde{\mathbf{C}}$, we have the following commutative diagram.

$$
\begin{tikzcd}
\widetilde{\mathbf{C}}(\tilde{L}(i\tilde{L}F \otimes_{\mathrm{Day}} H), G) \arrow[r, "\widetilde{\mathbf{C}}(\tilde{L}(\eta_F \otimes_{\mathrm{Day}} H), G)"] \arrow[d, "\cong"] & \widetilde{\mathbf{C}}(\tilde{L}(F \otimes_{\mathrm{Day}} H), G) \arrow[d, "\cong"] \\
\overline{\mathbf{C}}(i\tilde{L}F \otimes_{\mathrm{Day}} H, iG) \arrow[r, "\overline{\mathbf{C}}(\eta_F \otimes_{\mathrm{Day}} H, iG)"] \arrow[d, "\cong"] & \overline{\mathbf{C}}(F \otimes_{\mathrm{Day}} H, iG) \arrow[d, "\cong"] \\
\overline{\mathbf{C}}(i\tilde{L}F, H \multimap_{\mathrm{Day}} iG) \arrow[r, "\overline{\mathbf{C}}(\eta_F, H \multimap_{\mathrm{Day}} iG)"] \arrow[d, "\overline{\mathbf{C}}(i\tilde{L}F, \eta_{H \multimap_{\mathrm{Day}} iG})"'] & \overline{\mathbf{C}}(F, H \multimap_{\mathrm{Day}} iG) \arrow[d, "\overline{\mathbf{C}}(F, \eta_{H \multimap_{\mathrm{Day}} iG})"] \\
\overline{\mathbf{C}}(i\tilde{L}F, i\tilde{L}(H \multimap_{\mathrm{Day}} iG)) \arrow[r, "\overline{\mathbf{C}}(\eta_F, i\tilde{L}(H \multimap_{\mathrm{Day}} iG))"] \arrow[d, "\cong"] & \overline{\mathbf{C}}(F, i\tilde{L}(H \multimap_{\mathrm{Day}} iG)) \arrow[ddl, "\mathrm{Id}"] \\
\widetilde{\mathbf{C}}(\tilde{L}F, \tilde{L}(H \multimap_{\mathrm{Day}} iG)) \arrow[d, "\cong"] & \\
\widetilde{\mathbf{C}}(F, i\tilde{L}(H \multimap_{\mathrm{Day}} iG)) &
\end{tikzcd}
$$

The top two squares commute by naturality of the adjunctions, the third square commutes by the bifunctoriality of $\overline{\mathbf{C}}(-,-)$ and the bottom triangle commutes by properties of the adjunction $\tilde{L} \dashv i$. ☐

With the help of Theorem F.2, one can verify that $\tilde{L}$ is strong monoidal, e.g., for any $F, G \in \overline{\mathbf{C}}$,

$$\tilde{L}F \otimes_{\mathrm{Lam}} \tilde{L}G = \tilde{L}(i\tilde{L}F \otimes_{\mathrm{Day}} i\tilde{L}G) \cong \tilde{L}(F \otimes_{\mathrm{Day}} G).$$

# G Proof of Theorem 4.13

We write $\beta : i \circ \widetilde{T} \to \overline{T} \circ i$ to denote the isomorphism $i \circ \widetilde{T} \cong \overline{T} \circ i$.

**Theorem G.1.** *The following diagrams commute.*

1. *Suppose $F \in \widetilde{\mathbf{C}}$.*

$$
\begin{tikzcd}
iF \arrow[r, "i\eta^{\widetilde{T}}"] \arrow[dr, "\eta^{\overline{T}}"] & i\widetilde{T}F \arrow[d, "\beta"] \\
& \overline{T}iF
\end{tikzcd}
$$

2. *Suppose $F \in \overline{\mathbf{C}}$.*

$$
\begin{tikzcd}
\tilde{L}F \arrow[r, "\tilde{L}\eta^{\overline{T}}"] \arrow[dr, "\eta^{\widetilde{T}}"] & \tilde{L}\overline{T}F \arrow[d, "\rho"] \\
& \widetilde{T}\tilde{L}F
\end{tikzcd}
$$

3. *Suppose $F \in \widetilde{\mathbf{C}}$.*

$$\begin{tikzcd}
i\widetilde{T}\widetilde{T}F \arrow[r, "\beta"] \arrow[d, "i\mu^{\widetilde{T}}"] & \overline{T}i\widetilde{T}F \arrow[r, "\overline{T}\beta"] & \overline{T}\overline{T}iF \arrow[d, "\mu^{\overline{T}}"] \\
i\widetilde{T}F \arrow[rr, "\beta"] & & \overline{T}iF
\end{tikzcd}$$

4. *Suppose $F \in \overline{\mathbf{C}}$.*

$$\begin{tikzcd}
\tilde{L}\overline{T}\overline{T}F \arrow[r, "\rho"] \arrow[d, "\tilde{L}\mu^{\overline{T}}"] & \widetilde{T}\tilde{L}\overline{T}F \arrow[r, "\widetilde{T}\rho"] & \widetilde{T}\widetilde{T}\tilde{L}F \arrow[d, "\mu^{\widetilde{T}}"] \\
\tilde{L}\overline{T}F \arrow[rr, "\rho"] & & \widetilde{T}\tilde{L}F
\end{tikzcd}$$

5. *Suppose $F \in \widetilde{\mathbf{C}}$.*

$$\begin{tikzcd}
\tilde{L}i\widetilde{T}F \arrow[r, "\tilde{L}\beta"] \arrow[d, "\varepsilon"] & \tilde{L}\overline{T}iF \arrow[d, "\rho"] \\
\widetilde{T}F & \widetilde{T}\tilde{L}iF \arrow[l, "\widetilde{T}\varepsilon"]
\end{tikzcd}$$

6. *Suppose $F \in \overline{\mathbf{C}}, G \in \widetilde{\mathbf{C}}$.*

$$\begin{tikzcd}
\tilde{L}\overline{T}(i\widetilde{T}G \otimes_D F) \arrow[r, "\tilde{L}\overline{T}(\beta \otimes_D F)"] \arrow[d, "\rho"] & \tilde{L}\overline{T}(\overline{T}iG \otimes_D F) \arrow[r, "\tilde{L}\overline{T}\gamma"] & \tilde{L}\overline{T}(F \otimes_D \overline{T}iG) \arrow[d, "\rho"] \\
\widetilde{T}\tilde{L}(i\widetilde{T}G \otimes_D F) \arrow[r, "\widetilde{T}\tilde{L}\gamma"] & \widetilde{T}\tilde{L}(F \otimes_D i\widetilde{T}G) \arrow[r, "\widetilde{T}\tilde{L}(F \otimes_D \beta)"] & \widetilde{T}\tilde{L}(F \otimes_D \overline{T}iG)
\end{tikzcd}$$

7. *Suppose $F, G \in \overline{\mathbf{C}}$.*

$$\begin{tikzcd}
\tilde{L}(F \otimes_D \overline{T}G) \arrow[r, "\tilde{L}\bar{t}"] \arrow[d, "\tilde{L}(F \otimes_D \eta^{i\tilde{L}})"] & \tilde{L}\overline{T}(F \otimes_D G) \arrow[r, "\rho"] & \widetilde{T}\tilde{L}(F \otimes_D G) \arrow[r, "\widetilde{T}\tilde{L}(F \otimes_d \eta^{i\tilde{L}})"] & \widetilde{T}\tilde{L}(F \otimes_D i\tilde{L}G) \\
\tilde{L}(F \otimes_D i\tilde{L}\overline{T}G) \arrow[r, "\tilde{L}(F \otimes_D i\rho)"] & \tilde{L}(F \otimes_D i\widetilde{T}\tilde{L}G) \arrow[r, "\tilde{L}(F \otimes_D \beta)"] & \tilde{L}(F \otimes_D \overline{T}i\tilde{L}G) \arrow[r, "\tilde{L}\bar{t}"] & \tilde{L}\overline{T}(F \otimes_D i\tilde{L}G) \arrow[u, "\rho"]
\end{tikzcd}$$

*Proof.* 1. We have

$$\overline{\mathbf{C}}(iF, \overline{T}iF) = \overline{\mathbf{C}}(iF, \overline{\Delta}\overline{U_0}iF) \cong \overline{\mathbf{C}}(\overline{U_0}iF, \overline{U_0}iF) \cong \overline{\mathbf{C}}(j\overline{U_0}'F, j\overline{U_0}'F) \cong \widetilde{\mathbf{C}}(\overline{U_0}'F, \overline{U_0}'F) \cong \widetilde{\mathbf{C}}(F, \widetilde{T}F)$$

.

2. If we unfold the definition of $\rho$, we have the following diagram.

$$\begin{tikzcd}
\tilde{L}F \arrow[r, "\tilde{L}\eta^{\overline{T}}"] \arrow[dddr, bend right, "\eta^{\widetilde{T}}"'] & \tilde{L}\overline{T}F \arrow[d, "\tilde{L}\overline{T}\eta^{i\tilde{L}}"] \\
& \tilde{L}\overline{T}i\tilde{L}F \arrow[d, "\tilde{L}\beta^{-1}"] \\
& \tilde{L}i\widetilde{T}\tilde{L}F \arrow[d, "\varepsilon"] \\
& \widetilde{T}\tilde{L}F
\end{tikzcd}$$

Note that we have the following commutative diagram, by naturality and (1).

$$\begin{tikzcd}
F \arrow[r, "\eta^{\overline{T}}"] \arrow[d, "\eta^{\tilde{L}}"] & \overline{T}F \arrow[r, "\overline{T}\eta^{i\tilde{L}}"] & \overline{T}i\tilde{L}F \arrow[r, "\beta^{-1}"] & i\widetilde{T}\tilde{L}F \\
i\tilde{L}F \arrow[urr, "\eta^{\overline{T}}"] \arrow[urrr, "i\eta^{\widetilde{T}}"'] & & &
\end{tikzcd}$$

Therefore we just need to show the following diagram commutes (and indeed it does).

$$\begin{tikzcd}
\tilde{L}F \arrow[r, "\tilde{L}\eta^{i\tilde{L}}"] \arrow[dr, "\mathrm{Id}"] & \tilde{L}i\tilde{L}F \arrow[r, "\tilde{L}i\eta^{\widetilde{T}}"] \arrow[d, "\varepsilon"] & \tilde{L}i\widetilde{T}\tilde{L}F \arrow[d, "\varepsilon"] \\
& \tilde{L}F \arrow[r, "\eta^{\widetilde{T}}"] & \widetilde{T}\tilde{L}F
\end{tikzcd}$$

3. Since each component of $\mu$ and $\beta$ is an identity, the diagram commutes.
4. If we unfold the definition of $\rho$, we have the following diagram.

$$\begin{tikzcd}
\tilde{L}\overline{T}\overline{T}F \arrow[r, "\tilde{L}\overline{T}\eta^{i\tilde{L}}"] \arrow[d, "\tilde{L}\mu^{\overline{T}}"] & \tilde{L}\overline{T}i\tilde{L}\overline{T}F \arrow[r, "\tilde{L}\beta^{-1}"] & \tilde{L}i\widetilde{T}\tilde{L}\overline{T}F \arrow[r, "\varepsilon"] \arrow[d, "\tilde{L}i\widetilde{T}\tilde{L}\overline{T}\eta^{i\tilde{L}}"] & \widetilde{T}\tilde{L}\overline{T}F \arrow[d, "\widetilde{T}\tilde{L}\overline{T}\eta^{i\tilde{L}}"] \\
\tilde{L}\overline{T}F \arrow[d, "\tilde{L}\overline{T}\eta^{\tilde{L}}"] & & \tilde{L}i\widetilde{T}\tilde{L}\overline{T}i\tilde{L}F \arrow[r, "\varepsilon"] \arrow[d, "\tilde{L}i\widetilde{T}\tilde{L}\beta^{-1}"] & \widetilde{T}\tilde{L}\overline{T}i\tilde{L}F \arrow[d, "\widetilde{T}\tilde{L}\beta^{-1}"] \\
\tilde{L}\overline{T}i\tilde{L}F \arrow[d, "\tilde{L}\beta^{-1}"] & & \tilde{L}i\widetilde{T}\tilde{L}i\widetilde{T}\tilde{L}F \arrow[r, "\varepsilon"] \arrow[d, "\tilde{L}i\widetilde{T}\varepsilon"] & \widetilde{T}\tilde{L}i\widetilde{T}\tilde{L}F \arrow[d, "\widetilde{T}\varepsilon"] \\
\tilde{L}i\widetilde{T}\tilde{L}F \arrow[d, "\varepsilon"] & & \tilde{L}i\widetilde{T}\widetilde{T}\tilde{L}F \arrow[ll, "\tilde{L}i\mu^{\widetilde{T}}"] \arrow[r, "\varepsilon"] & \widetilde{T}\widetilde{T}\tilde{L}F \arrow[dlll, bend left, "\mu^{\widetilde{T}}"] \\
\widetilde{T}\tilde{L}F & & &
\end{tikzcd}$$

All of the squares commute by naturality. We just need to show the left diagram commutes. It commutes because the following commutes.

$$\begin{tikzcd}
\overline{T}\overline{T}F \arrow[r, "\overline{T}\eta^{i\tilde{L}}"] \arrow[d, "\mu"] \arrow[dr, "\overline{T}\overline{T}\eta^{i\tilde{L}}"'] & \overline{T}i\tilde{L}\overline{T}F \arrow[rr, "\beta^{-1}"] \arrow[dr, "\overline{T}i\tilde{L}\overline{T}\eta^{i\tilde{L}}"'] & & i\widetilde{T}\tilde{L}\overline{T}F \arrow[dd, "i\tilde{L}\overline{T}\eta^{i\tilde{L}}"] \\
\overline{T}F \arrow[d, "\overline{T}\eta^{i\tilde{L}}"] & \overline{T}\overline{T}i\tilde{L}F \arrow[r, "\overline{T}\eta^{i\tilde{L}}"] \arrow[dl, "\mu"] \arrow[d, "\overline{T}\beta^{-1}"] & \overline{T}i\tilde{L}\overline{T}i\tilde{L}F \arrow[d, "\overline{T}i\tilde{L}\beta^{-1}"] \arrow[dr, "\beta^{-1}"] & \\
\overline{T}i\tilde{L}F \arrow[d, "\beta^{-1}"] & \overline{T}i\widetilde{T}\tilde{L}F \arrow[r, "\overline{T}\eta^{i\tilde{L}}"] \arrow[d, "\beta^{-1}"] & \overline{T}i\tilde{L}i\widetilde{T}\tilde{L}F \arrow[dr, "\beta^{-1}"] & i\widetilde{T}\tilde{L}\overline{T}i\tilde{L}F \arrow[d, "i\widetilde{T}\tilde{L}\beta^{-1}"] \\
i\widetilde{T}\tilde{L}F & i\widetilde{T}\widetilde{T}\tilde{L}F \arrow[l, "i\mu"] & & i\widetilde{T}\tilde{L}i\widetilde{T}\tilde{L}F \arrow[ll, "i\widetilde{T}\varepsilon"]
\end{tikzcd}$$

The above diagram commutes, because all the squares commutes by naturality. The bottom left corner diagram commutes by (3). Note that $i\varepsilon \circ \eta^{i\tilde{L}} = \mathrm{Id}$.

5. If we unfold the definition of $\rho$, we have the following diagram.

$$\begin{tikzcd}
\tilde{L}i\widetilde{T}F \arrow[r, "\tilde{L}\beta"] \arrow[d, "\tilde{L}\beta"] & \tilde{L}\overline{T}iF \arrow[d, "\tilde{L}\overline{T}\eta^{i\tilde{L}}"] \\
\tilde{L}\overline{T}iF \arrow[d, "\tilde{L}\beta^{-1}"] & \tilde{L}\overline{T}i\tilde{L}iF \arrow[l, "\tilde{L}\overline{T}i\varepsilon"] \arrow[d, "\tilde{L}\beta^{-1}"] \\
\tilde{L}i\widetilde{T}F \arrow[d, "\varepsilon"] & \tilde{L}i\widetilde{T}\tilde{L}iF \arrow[l, "\tilde{L}i\widetilde{T}\varepsilon"] \arrow[d, "\varepsilon"] \\
\widetilde{T}F & \widetilde{T}\tilde{L}iF \arrow[l, "\widetilde{T}\varepsilon"]
\end{tikzcd}$$

The bottom two squares commute by naturality. The top square commutes because $i\varepsilon \circ \eta^{i\tilde{L}} = \mathrm{Id}$.

6. See the following commutative diagram.

$$\begin{tikzcd}
\tilde{L}\overline{T}(i\widetilde{T}G \otimes_D F) \arrow[r, "\tilde{L}\overline{T}(\beta\otimes_D F)"] \arrow[dd, "\rho"] & \tilde{L}\overline{T}(\overline{T}iG \otimes_D F) \arrow[r, "\tilde{L}\overline{T}\gamma"] \arrow[d, "\rho"] & \tilde{L}\overline{T}(F \otimes_D \overline{T}iG) \arrow[dd, "\rho"] \\
& \widetilde{T}\tilde{L}(\overline{T}iG \otimes_D F) \arrow[dr, "\widetilde{T}\tilde{L}\gamma"] & \\
\widetilde{T}\tilde{L}(i\widetilde{T}G \otimes_D F) \arrow[r, "\widetilde{T}\tilde{L}\gamma"] \arrow[ur, "\widetilde{T}\tilde{L}(\beta\otimes_D F)"] & \widetilde{T}\tilde{L}(F \otimes_D i\widetilde{T}G) \arrow[r, "\widetilde{T}\tilde{L}(F\otimes_D\beta)"] & \widetilde{T}\tilde{L}(F \otimes_D \overline{T}iG)
\end{tikzcd}$$

7. Consider the following diagram.

$$\begin{tikzcd}
\tilde{L}(F \otimes_D \overline{T}G) \arrow[r, "\tilde{L}\bar{t}"] \arrow[d, "\tilde{L}(F\otimes\eta^{i\tilde{L}})"] \arrow[drr, "\tilde{L}(F\otimes\overline{T}\eta^{i\tilde{L}})"] & \tilde{L}\overline{T}(F \otimes_D G) \arrow[r, "\rho"] \arrow[drr, "\tilde{L}\overline{T}(F\otimes\eta^{i\tilde{L}})"] & \widetilde{T}\tilde{L}(F \otimes_D G) \arrow[r, "\widetilde{T}\tilde{L}(F\otimes\eta^{i\tilde{L}})"] & \widetilde{T}\tilde{L}(F \otimes_D i\tilde{L}G) \\
\tilde{L}(F \otimes_D i\tilde{L}\overline{T}G) \arrow[r, "\tilde{L}(F\otimes_D i\rho)"] & \tilde{L}(F \otimes_D i\widetilde{T}\tilde{L}G) \arrow[r, "\tilde{L}(F\otimes_D\beta)"'] & \tilde{L}(F \otimes_D \overline{T}i\tilde{L}G) \arrow[r, "\tilde{L}\bar{t}"] & \tilde{L}\overline{T}(F \otimes_D i\tilde{L}G) \arrow[u, "\rho"]
\end{tikzcd}$$

Note that the very right diagram and the middle square commute by naturality. The left diagram commutes because the following diagram commutes (with $\rho$ unfolded).

$$
\begin{tikzcd}
\tilde{L}(F \otimes_D \overline{T}G) \arrow[rr, "\tilde{L}(F\otimes\overline{T}\eta^{i\tilde{L}})"] \arrow[d, "\tilde{L}(F\otimes\eta^{i\tilde{L}})"'] \arrow[dr, "\tilde{L}(F\otimes\overline{T}\eta^{i\tilde{L}})"] & & \tilde{L}(F \otimes_D \overline{T}i\tilde{L}G) \\
\tilde{L}(F \otimes_D i\tilde{L}\overline{T}G) \arrow[d, "\tilde{L}(F\otimes i\tilde{L}\overline{T}\eta^{i\tilde{L}})"'] & \tilde{L}(F \otimes_D \overline{T}i\tilde{L}G) \arrow[ur, "\mathrm{Id}"] \arrow[r, "\tilde{L}(F\otimes_D\beta^{-1})"] \arrow[dl, "\tilde{L}(F\otimes\eta^{i\tilde{L}})"'] & \tilde{L}(F \otimes_D i\widetilde{T}\tilde{L}G) \arrow[dl, "\tilde{L}(F\otimes\eta^{i\tilde{L}})"'] \\
\tilde{L}(F \otimes_D i\tilde{L}\overline{T}i\tilde{L}G) \arrow[r, "\tilde{L}(F\otimes_D i\tilde{L}\beta^{-1})"'] & \tilde{L}(F \otimes_D i\tilde{L}i\widetilde{T}\tilde{L}G) \arrow[r, "\tilde{L}(F\otimes_D i\varepsilon)"'] & \tilde{L}(F \otimes_D i\widetilde{T}\tilde{L}G) \arrow[uu, bend right, "\tilde{L}(F\otimes_D\beta)"']
\end{tikzcd}
$$

□

**Theorem G.2.** *The $\mathscr{V}$-functor $\widetilde{T}$ is a commutative strong monad. The strength is given by $\tilde{t}_{F,G} : F \otimes_{\mathrm{Lam}} \widetilde{T}G \to \widetilde{T}(F \otimes_{\mathrm{Lam}} G)$ for any $F, G \in \widetilde{\mathbf{C}}$.*

*Proof.* For any $F, G \in \widetilde{\mathbf{C}}$, we define $\tilde{t}_{F,G}$ by the following composition.

$$
\begin{aligned}
F \otimes_{\mathrm{Lam}} \widetilde{T}G &= \tilde{L}(iF \otimes_{\mathrm{Day}} i\widetilde{T}G) \\
&\overset{\beta}{\cong} \tilde{L}(iF \otimes_{\mathrm{Day}} \overline{T}iG) \xrightarrow{\bar{t}} \tilde{L}(\overline{T}(iF \otimes_{\mathrm{Day}} iG)) \xrightarrow{\rho} \widetilde{T}(\tilde{L}(iF \otimes_{\mathrm{Day}} iG)) = \widetilde{T}(F \otimes_{\mathrm{Lam}} G).
\end{aligned}
$$

Now we need to show $\widetilde{T}$ is a commutative strong monad.

•

$$
\begin{tikzcd}
F \otimes_{\mathrm{Lam}} G \arrow[r, "\mathrm{Id}_F \otimes_{\mathrm{Lam}} \eta"] \arrow[d, "\eta"] & F \otimes_{\mathrm{Lam}} \widetilde{T}G \arrow[dl, "\tilde{t}"] \\
\widetilde{T}(F \otimes_{\mathrm{Lam}} G) &
\end{tikzcd}
$$

The above diagram commutes because the following diagram commutes (by properties of $\bar{t}$ and Theorem G.1(1)+(2)).

$$
\begin{tikzcd}
\tilde{L}(iF \otimes_{\mathrm{Day}} iG) \arrow[r, "\tilde{L}(\mathrm{Id}\otimes_{\mathrm{D}} i\eta^{\widetilde{T}})"] \arrow[d, "\eta^{\widetilde{T}}"] \arrow[rr, bend right, "\tilde{L}(\mathrm{Id}\otimes_{\mathrm{D}}\eta^{\overline{T}})"] \arrow[drr, bend right, "\tilde{L}\eta^{\overline{T}}"'] & \tilde{L}(iF \otimes_{\mathrm{Day}} i\widetilde{T}G) \arrow[r, "\tilde{L}(\mathrm{Id}\otimes_{\mathrm{D}}\beta)"] & \tilde{L}(iF \otimes_{\mathrm{Day}} \overline{T}iG) \arrow[d, "\tilde{L}\bar{t}"] \\
\widetilde{T}\tilde{L}(iF \otimes_{\mathrm{Day}} iG) & & \tilde{L}\overline{T}(iF \otimes_{\mathrm{Day}} iG) \arrow[ll, bend left, "\rho"]
\end{tikzcd}
$$

•

$$
\begin{tikzcd}
F \otimes_{\mathrm{Lam}} \widetilde{T}\widetilde{T}G \arrow[r, "\tilde{t}"] \arrow[d, "F\otimes_{\mathrm{Lam}}\mu^{\widetilde{T}}"] & \widetilde{T}(F \otimes_{\mathrm{Lam}} \widetilde{T}G) \arrow[r, "\widetilde{T}\tilde{t}"] & \widetilde{T}\widetilde{T}(F \otimes_{\mathrm{Lam}} G) \arrow[d, "\mu^{\widetilde{T}}"] \\
F \otimes_{\mathrm{Lam}} \widetilde{T}G \arrow[rr, "\tilde{t}"] & & \widetilde{T}(F \otimes_{\mathrm{Lam}} G)
\end{tikzcd}
$$

The above diagram commutes because the following diagram commutes (all the inner diagrams commute, by naturality, properties of $\bar{t}$, and Theorem G.1(3)+(4)).

$$
\begin{tikzcd}
\tilde{L}(iF \otimes_D i\widetilde{T}\widetilde{T}G) \arrow[r, "\tilde{L}(\mathrm{Id}\otimes_D\beta)"] \arrow[d, "\tilde{L}(\mathrm{Id}\otimes_D i\mu^{\widetilde{T}})"] & \tilde{L}(iF \otimes_D \overline{T}i\widetilde{T}G) \arrow[r, "\tilde{L}\bar{t}"] \arrow[d, "\tilde{L}(\mathrm{Id}\otimes_D\overline{T}\beta)"] & \tilde{L}\overline{T}(iF \otimes_D i\widetilde{T}G) \arrow[d, "\rho"] \arrow[ddl, "\tilde{L}\overline{T}(\mathrm{Id}\otimes_D\beta)"] \\
\tilde{L}(iF \otimes_D i\widetilde{T}G) \arrow[d, "\tilde{L}(\mathrm{Id}\otimes\beta)"] & \tilde{L}(iF \otimes_D \overline{T}\overline{T}iG) \arrow[d, "\tilde{L}\bar{t}"] \arrow[dl, "\tilde{L}(\mathrm{Id}\otimes_D\mu^{\overline{T}})"] & \widetilde{T}\tilde{L}(iF \otimes_D i\widetilde{T}G) \arrow[d, "\widetilde{T}\tilde{L}(\mathrm{Id}\otimes_D\beta)"] \\
\tilde{L}(iF \otimes_D \overline{T}iG) \arrow[d, "\tilde{L}\bar{t}"] & \tilde{L}\overline{T}(iF \otimes_D \overline{T}iG) \arrow[r, "\rho"] \arrow[d, "\tilde{L}\overline{T}\bar{t}"] & \widetilde{T}\tilde{L}(iF \otimes_D \overline{T}iG) \arrow[d, "\widetilde{T}\tilde{L}\bar{t}"] \\
\tilde{L}\overline{T}(iF \otimes_D iG) \arrow[d, "\rho"] & \tilde{L}\overline{T}\overline{T}(iF \otimes_D iG) \arrow[l, "\tilde{L}\mu^{\overline{T}}"'] \arrow[r, "\rho"] & \widetilde{T}\tilde{L}\overline{T}(iF \otimes_D iG) \arrow[d, "\widetilde{T}\rho"] \\
\widetilde{T}\tilde{L}(iF \otimes_D iG) & & \widetilde{T}\widetilde{T}\tilde{L}(iF \otimes_D iG) \arrow[ll, "\mu^{\widetilde{T}}"']
\end{tikzcd}
$$

- 

$$
\begin{tikzcd}
(F \otimes_{\mathrm{Lam}} G) \otimes_{\mathrm{Lam}} \widetilde{T}H \arrow[rr, "\tilde{t}"] \arrow[d, "\alpha"] & & \widetilde{T}((F \otimes_{\mathrm{Lam}} G) \otimes_{\mathrm{Lam}} H) \arrow[d, "\widetilde{T}\alpha"] \\
F \otimes_{\mathrm{Lam}} (G \otimes_{\mathrm{Lam}} \widetilde{T}H) \arrow[r, "\mathrm{Id}_F \otimes_{\mathrm{Lam}} \tilde{t}"] & F \otimes_{\mathrm{Lam}} \widetilde{T}(G \otimes_{\mathrm{Lam}} H) \arrow[r, "\tilde{t}"] & \widetilde{T}(F \otimes_{\mathrm{Lam}} (G \otimes_{\mathrm{Lam}} H))
\end{tikzcd}
$$

Again, the above diagram commutes because the following diagram commutes (all of the inner diagrams commute by naturality, properties of $\bar{t}$, and Theorem G.1(7)).

$$
\begin{tikzcd}
\tilde{L}(i\tilde{L}(iF \otimes_D iG) \otimes_D i\widetilde{T}H) \arrow[r, "\tilde{L}(\mathrm{Id}\otimes_D\beta)"] \arrow[d, "(\tilde{L}(\eta^{i\tilde{L}}\otimes_D\mathrm{Id}))^{-1}"] & \tilde{L}(i\tilde{L}(iF \otimes_D iG) \otimes_D \overline{T}iH) \arrow[rr, "\tilde{L}\bar{t}"] \arrow[d, "(\tilde{L}(\eta^{i\tilde{L}}\otimes_D\mathrm{Id}))^{-1}"] & & \tilde{L}\overline{T}(i\tilde{L}(iF \otimes_D iG) \otimes_D iH) \arrow[d, "\rho"] \\
\tilde{L}((iF \otimes_D iG) \otimes_D i\widetilde{T}H) \arrow[r, "\tilde{L}(\mathrm{Id}\otimes_D\beta)"] \arrow[d, "\tilde{L}\alpha"] & \tilde{L}((iF \otimes_D iG) \otimes_D \overline{T}iH) \arrow[r, "\tilde{L}\bar{t}"] \arrow[d, "\tilde{L}\alpha"] & \tilde{L}\overline{T}((iF \otimes_D iG) \otimes_D iH) \arrow[dr, "\rho"] \arrow[dddl, bend left, "\tilde{L}\overline{T}\alpha"] & \widetilde{T}\tilde{L}(i\tilde{L}(iF \otimes_D iG) \otimes_D iH) \arrow[d, "\widetilde{T}(\tilde{L}(\eta^{i\tilde{L}}\otimes_D\mathrm{Id}))^{-1}"] \\
\tilde{L}(iF \otimes_D (iG \otimes_D i\widetilde{T}H)) \arrow[r, "\tilde{L}(\mathrm{Id}\otimes_D(\mathrm{Id}\otimes\beta))"] \arrow[d, "\tilde{L}(\mathrm{Id}\otimes_D\eta^{i\tilde{L}})"] & \tilde{L}(iF \otimes_D (iG \otimes_D \overline{T}iH)) \arrow[d, "\tilde{L}(\mathrm{Id}\otimes_D\bar{t})"] \arrow[ddl, "\tilde{L}(\mathrm{Id}\otimes_D\eta^{i\tilde{L}})"'] & & \widetilde{T}\tilde{L}((iF \otimes_D iG) \otimes_D iH) \arrow[d, "\widetilde{T}\tilde{L}\alpha"] \\
\tilde{L}(iF \otimes_D i\tilde{L}(iG \otimes_D i\widetilde{T}H)) \arrow[d, "\tilde{L}(\mathrm{Id}\otimes_D i\tilde{L}(\mathrm{Id}\otimes_D\beta))"'] & \tilde{L}(iF \otimes_D \overline{T}(iG \otimes_D iH)) \arrow[d, "\tilde{L}\bar{t}"] \arrow[ddl, "\tilde{L}(\mathrm{Id}\otimes_D\eta^{i\tilde{L}})"] & & \widetilde{T}\tilde{L}(iF \otimes_D (iG \otimes_D iH)) \arrow[d, "\widetilde{T}(\tilde{L}(\mathrm{Id}\otimes_D\eta^{i\tilde{L}}))"] \\
\tilde{L}(iF \otimes_D i\tilde{L}(iG \otimes_D \overline{T}iH)) \arrow[d, "\tilde{L}(\mathrm{Id}\otimes_D i\tilde{L}\bar{t})"'] & \tilde{L}\overline{T}(iF \otimes_D (iG \otimes_D iH)) \arrow[urr, "\rho"] & & \widetilde{T}\tilde{L}(iF \otimes_D i\tilde{L}(iG \otimes_D iH)) \\
\tilde{L}(iF \otimes_D i\tilde{L}\overline{T}(iG \otimes_D iH)) \arrow[r, "\tilde{L}(\mathrm{Id}\otimes_D i\rho)"] & \tilde{L}(iF \otimes_D i\widetilde{T}\tilde{L}(iG \otimes_D iH)) \arrow[r, "\tilde{L}(\mathrm{Id}\otimes_D\beta)"] & \tilde{L}(iF \otimes_D \overline{T}i\tilde{L}(iG \otimes_D iH)) \arrow[r, "\tilde{L}\bar{t}"] & \tilde{L}\overline{T}(iF \otimes_D i\tilde{L}(iG \otimes_D iH)) \arrow[u, "\rho"]
\end{tikzcd}
$$

- 

$$
\begin{tikzcd}
I \otimes_{\mathrm{Lam}} \widetilde{T}F \arrow[r, "\tilde{t}_{I,F}"] \arrow[dr, "\lambda_{\widetilde{T}F}"'] & \widetilde{T}(I \otimes_{\mathrm{Lam}} F) \arrow[d, "\widetilde{T}\lambda_F"] \\
& \widetilde{T}F
\end{tikzcd}
$$

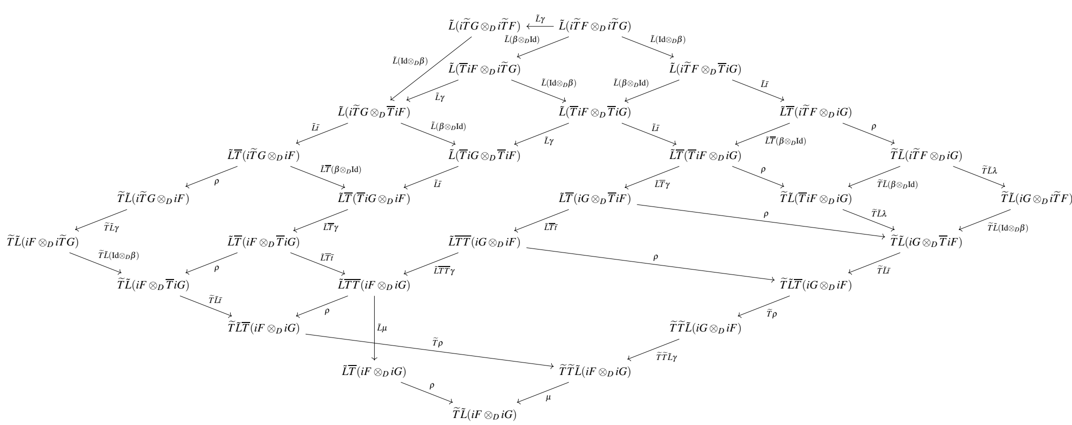
The above diagram commutes because the following commutes (all the inner diagrams commute, by naturality, property of $\bar{t}$, and Theorem G.1 (5)).

$$\begin{tikzcd}
\tilde{L}(I \otimes_D i\widetilde{T}F) \arrow[r, "\tilde{L}(\mathrm{Id}\otimes_D \beta)"] \arrow[d, "\tilde{L}\lambda"] & \tilde{L}(I \otimes_D \overline{T}iF) \arrow[r, "\tilde{L}\bar{t}"] \arrow[d, "\tilde{L}\lambda"] & \tilde{L}\overline{T}(I \otimes_D iF) \arrow[dl, "\tilde{L}\overline{T}\lambda"'] \arrow[d, "\rho"] \\
\tilde{L}i\widetilde{T}F \arrow[r, "\tilde{L}\beta"] \arrow[d, "\varepsilon"] & \tilde{L}\overline{T}iF \arrow[dr, "\rho"] & \widetilde{T}\tilde{L}(I \otimes_D iF) \arrow[d, "\widetilde{T}\tilde{L}\lambda"] \\
\widetilde{T}F & & \widetilde{T}\tilde{L}iF \arrow[ll, "\widetilde{T}\varepsilon"]
\end{tikzcd}$$

- Lastly, since $\widetilde{\mathbf{C}}$ is symmetric monoidal with $\gamma_{F,G} : F \otimes_{\mathrm{Lam}} G \to G \otimes_{\mathrm{Lam}} F$. We define the costrength $\sigma_{F,G} := \widetilde{T}\gamma_{G,F} \circ \tilde{t}_{G,F} \circ \gamma_{\widetilde{T}F,G} : \widetilde{T}F \otimes_{\mathrm{Lam}} G \to \widetilde{T}(F \otimes_{\mathrm{Lam}} G)$. We need to show the following diagram commutes.

$$\begin{tikzcd}
 & \widetilde{T}F \otimes_{\mathrm{Lam}} \widetilde{T}G \arrow[dl, "\sigma"'] \arrow[dr, "\tilde{t}"] & \\
\widetilde{T}(F \otimes_{\mathrm{Lam}} \widetilde{T}G) \arrow[d, "\widetilde{T}\tilde{t}"] & & \widetilde{T}(\widetilde{T}F \otimes_{\mathrm{Lam}} G) \arrow[d, "\widetilde{T}\sigma"] \\
\widetilde{T}\widetilde{T}(F \otimes_{\mathrm{Lam}} G) \arrow[dr, "\mu^{\widetilde{T}}"] & & \widetilde{T}\widetilde{T}(F \otimes_{\mathrm{Lam}} G) \arrow[dl, "\mu^{\widetilde{T}}"] \\
 & \widetilde{T}(F \otimes_{\mathrm{Lam}} G) &
\end{tikzcd}$$

Again, the above diagram commutes because the following diagram commutes (all the inner diagrams commute, by naturality, properties of $\bar{t}$, and Theorem G.1(4)+(6)).



□

## H Proof of Theorem 4.17

**Theorem H.1.** *The $\mathscr{V}$-category $\widetilde{\mathbf{C}}$ is a model for Proto-Quipper with dynamic lifting.*

*Proof.* We have already shown that $\widetilde{\mathbf{C}}$ satisfies conditions **a–e** and **g–h**. In the following we will focus on condition **f**.

- First we need to define a functor $\psi : \mathbf{M} \hookrightarrow V(\widetilde{\mathbf{C}})$. We define it as the following composition.

$$\mathbf{M} \cong V(\mathbf{C}) \overset{V\overline{y}}{\hookrightarrow} V(\widetilde{\mathbf{C}})$$

  The functor $\psi$ is strong monoidal, because $V\overline{y}$ is strong monoidal.

- Next we need to define a functor $\phi : \mathbf{Q} \hookrightarrow Kl_{V\widetilde{T}}(V(\widetilde{\mathbf{C}}))$. We write

$$\theta_{F,G} : \widetilde{\mathbf{C}}(F, \tilde{T}G) \cong [\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}(\overline{U_0}'F, \overline{U_0}'G)$$

  for any $F, G \in \widetilde{\mathbf{C}}$. We also write $\Omega : [\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}} \cong [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$. We have

$$Kl_{V\widetilde{T}}(V(\widetilde{\mathbf{C}}))(F,G) = \mathscr{V}(1, \widetilde{\mathbf{C}}(F, \tilde{T}G))$$

$$\overset{\mathscr{V}(1,\theta_{F,G})}{\cong} \mathscr{V}(1, [\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}(\overline{U_0}'F, \overline{U_0}'G)) \overset{\Omega_{\overline{U_0}'F,\overline{U_0}'G}}{\cong} [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}(F^0, G^0)$$

  for any $F, G \in \widetilde{\mathbf{C}}$. The category $Kl_{V\widetilde{T}}(V(\widetilde{\mathbf{C}}))$ is enriched in convex spaces because $[\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}$ is enriched in convex spaces.

  Now we define $\phi$. On objects, we define

$$\phi(S) = \overline{y}S = \mathbf{C}(-,S).$$

  On morphisms, for any $S, U \in \mathbf{Q}$, we define $\phi_{S,U}$ by the following composition of isomorphisms.

$$\mathbf{Q}(S,U) \overset{\kappa_{S,U}}{\cong} [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}(\kappa S, \kappa U) = [\mathbf{Set}^{\mathbf{Q}^{\mathrm{op}}}]_{\mathrm{prod}}((\overline{y}S)^0, (\overline{y}U)^0)$$

$$\overset{\Omega^{-1}_{\overline{U_0}'\overline{y}S,\overline{U_0}'\overline{y}U}}{\cong} \mathscr{V}(1, [\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}(\overline{U_0}'\overline{y}S, \overline{U_0}'\overline{y}U)) \overset{\mathscr{V}(1,\theta^{-1}_{\overline{y}S,\overline{y}U})}{\cong} Kl_{V\widetilde{T}}(V(\widetilde{\mathbf{C}}))(\overline{y}S, \overline{y}U)$$

  Since the Lambek embedding $\kappa_{S,U}$ preserves convex sum (Theorem 4.16) and the composition $\mathscr{V}(1,\theta^{-1}_{\overline{y}S,\overline{y}U}) \circ \Omega^{-1}_{\overline{U_0}'\overline{y}S,\overline{U_0}'\overline{y}U}$ preserves convex sum, we conclude that $\phi$ preserves convex sum.

  Next we need to show $\phi$ is strong monoidal. Since $\kappa$ is strong monoidal, we have the natural isomorphisms $I \overset{e'}{\cong} \kappa I$ and $\kappa S \otimes \kappa U \overset{m'_{S,U}}{\cong} \kappa(S \otimes U)$ for any $S, U \in \mathbf{Q}$. Recall that for any $S \in \mathbf{Q}$, $\kappa S = \mathbf{Q}(-,S) = \overline{U_0}'\mathbf{C}(-,S)$ and $\overline{U_0}$ is strong monoidal. Via the isomorphism $\Omega$ (which preserves the monoidal structure) we have the following natural isomorphisms in $[\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}]_{\mathrm{prod}}$.

$$\Omega^{-1}(e') : \overline{U_0}'\mathbf{C}(-,I) \to \overline{U_0}'\mathbf{C}(-,I)$$

$$\Omega^{-1}(m'_{S,U}) : \overline{U_0}'(\mathbf{C}(-,S) \otimes \mathbf{C}(-,U)) \to \overline{U_0}'\mathbf{C}(-,S \otimes U).$$

  Now let $m_{S,U} = \theta^{-1}_{\overline{y}S,\overline{y}U}(\Omega^{-1}(m'_{S,U})) : \mathbf{C}(-,S) \otimes \mathbf{C}(-,U) \to \widetilde{T}\mathbf{C}(-,S \otimes U)$ and $e = \theta^{-1}_{\overline{y}I,\overline{y}I}(\Omega^{-1}(e')) : \mathbf{C}(-,I) \to \widetilde{T}\mathbf{C}(-,I)$. It is obvious that $e$ is an isomorphism in $Kl_{V\widetilde{T}}(V(\widetilde{\mathbf{C}}))$. The inverse of $m_{S,U}$ is defined as $\theta^{-1}_{\overline{y}S,\overline{y}U}(\Omega^{-1}({m'_{S,U}}^{-1}))$, which can be verified. We can furthermore show that $m_{S,U}$ is natural and that $e, m_{S,U}$ satisfies the strength diagrams for any $S, U \in \mathbf{Q}$. For example, showing $m_{S,U}$ is natural in $Kl_{V\widetilde{T}}(V(\widetilde{\mathbf{C}}))$, via the adjunction $\overline{U_0}' \dashv \overline{\Delta}'$, is equivalent to the naturality of $\Omega^{-1}(m'_{S,U})$.

- Lastly, we want to show that the following diagram commutes.

$$\begin{array}{ccc} \mathbf{M}(S,U) & \xrightarrow{\psi_{S,U}} & \mathscr{V}(1,\widetilde{\mathbf{C}}(\overline{y}S,\overline{y}U)) = V(\widetilde{\mathbf{C}})(\overline{y}S,\overline{y}U) \\ \downarrow{\scriptstyle J_{S,U}} & & \downarrow{\scriptstyle E_{S,U}} \\ \mathbf{Q}(S,U) & \xleftarrow[\phi^{-1}_{S,U}]{} & \mathscr{V}(1,\widetilde{\mathbf{C}}(\overline{y}S,\widetilde{T}\overline{y}U)) = Kl_{V\widetilde{T}}(V(\widetilde{\mathbf{C}})(\overline{y}S,\overline{y}U)) \end{array}$$

Let $f \in \mathbf{M}(S,U)$. We write $(f, J_{S,U} f)$ for the corresponding map in $\mathscr{V}(1,\mathbf{C}(S,U))$. It corresponds to the following map in $\widetilde{\mathbf{C}}$ via the enriched Yoneda embedding.

$$\mathbf{C}(-,S) \xrightarrow{\overline{y}(f,J_{S,U}f)} \mathbf{C}(-,U)$$

Applying $\eta_U$ to the above map, we have the following.

$$\mathbf{C}(-,S) \xrightarrow{\overline{y}(f,J_{S,U}f)} \mathbf{C}(-,U) \xrightarrow{\eta_U} \widetilde{T}\mathbf{C}(-,U)$$

So for any $A \in \mathbf{C}$, we have the following.

$$\begin{array}{ccccc} \mathbf{M}(A,S) & \xrightarrow{\mathbf{M}(A,f)} & \mathbf{M}(A,U) & \xrightarrow{J_{A,U}} & \mathbf{Q}(A,U) \\ \downarrow{\scriptstyle J_{A,S}} & & \downarrow{\scriptstyle J_{A,U}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathbf{Q}(A,S) & \xrightarrow{\mathbf{Q}(A,J_{S,U}f)} & \mathbf{Q}(A,U) & \xrightarrow{\mathrm{Id}} & \mathbf{Q}(A,U) \end{array}$$

Since $\phi^{-1}_{S,U} = \kappa^{-1}_{S,U} \circ \Omega_{\overline{U_0}'\overline{y}S,\overline{U_0}'\overline{y}U} \circ \mathscr{V}(1,\theta_{\overline{y}S,\overline{y}U})$, we have

$$\phi^{-1}_{S,U}(\eta_U \circ \overline{y}(f,J_{S,U}f)) = \kappa^{-1}_{S,U}\Omega_{\overline{U_0}'\overline{y}S,\overline{U_0}'\overline{y}U}\mathscr{V}(1,\theta_{\overline{y}S,\overline{y}U})(\eta_U \circ \overline{y}(f,J_{S,U}f))$$

$$= \kappa^{-1}_{S,U}\Omega_{\overline{U_0}'\overline{y}S,\overline{U_0}'\overline{y}U}(\overline{U_0}'\overline{y}(f,J_{S,U}f)) = \kappa^{-1}_{S,U}(\mathbf{Q}(-,J_{S,U}f)) = J_{S,U}f \in \mathbf{Q}(S,U).$$

$\square$