There are infinitely many monotone games over L_5

Eric Demer, UCLA
Peter Selinger, Dalhousie University

Abstract

A notion of combinatorial game over a partially ordered set of atomic outcomes was recently introduced by Selinger. These games are appropriate for describing the value of positions in Hex and other monotone set coloring games. It is already known that there are infinitely many distinct monotone game values when the poset of atoms is not linearly ordered, and that there are only finitely many such values when the poset of atoms is linearly ordered with 4 or fewer elements. In this short paper, we settle the remaining case: when the atom poset has 5 or more elements, there are infinitely many distinct monotone values.

1 Introduction

Combinatorial game theory, introduced by Conway [2] and Berlekamp, Conway, and Guy [1] in the 1970s and 1980s, is a mathematical theory of sequential perfect information games. In its original form, this theory deals with games following the *normal play* convention, under which the last player who is able to make a move wins the game. However, combinatorial game theory has also been applied to many other situations, including *misère play*, in which the last player to move loses, as well as *scoring games*, in which the final outcome is a numerical score.

In [4], a new variant of scoring games was introduced in which the atomic positions are elements of a partially ordered set (poset). These games are appropriate for analyzing monotone set coloring games such as Hex, and in fact, they capture that class of games exactly [3]. In [4], it was shown that when the atom poset A is not linearly ordered, i.e., when it has a pair of incomparable elements, then there are infinitely many non-equivalent monotone game values over A, and when A is linearly ordered with 4 or fewer elements, there are only finitely many such values up to equivalence. It was stated in [4] without proof that when A is the 6-element linear order, there are infinitely many values, and it was conjectured that this is also true when A is the 5-element linear order.

The purpose of this short paper is to supply a positive answer to this conjecture.

2 Background

We briefly recall the definition of games over a poset and some of their properties. Full details can be found in [4]. Let A be a partially ordered set whose elements we call *atoms*. The class of combinatorial games over A is inductively defined as follows:

- For every atom $a \in A$, $[a]$ is a game, and
- Whenever L and R are non-empty sets of games, then $\{L \mid R\}$ is a game.

The fact that it is an inductive definition implies that there are no other games except the ones constructed above. A game of the form $[a]$ is called *atomic*, and we often write a instead of $[a]$ when no confusion arises. A game of the form $\{L \mid R\}$ is called *composite*. In the game $G = \{L \mid R\}$, the elements of L and R are called the *left options* and *right options* of G, respectively. The idea is that there are two players, called Left and Right, and in the game $G = \{L \mid R\}$, L represents the set of all moves available to Left, and R

represents the set of all moves available to Right. We use the usual notations of combinatorial game theory. Specifically, if $L = \{G_1, \ldots, G_n\}$ and $R = \{H_1, \ldots, H_m\}$, we write $\{G_1, \ldots, G_n \mid H_1, \ldots, H_m\}$ for $\{L \mid R\}$. We also write G^L and G^R for a typical left and right option of G. A *position* of a game G is either G itself, or an option of G, or an option of an option, and so on recursively.

On the class of games over a poset A, we define the relations $\leqslant$ and $\lhd$ by mutual recursion as follows:

- $G \leqslant H$ if all three of the following conditions hold:
  1. All left options G^L satisfy $G^L \lhd H$, and
  2. all right options H^R satisfy $G \lhd H^R$, and
  3. if G or H is atomic, then $G \lhd H$.
- $G \lhd H$ if at least one of the following conditions holds:
  1. There exists a right option G^R such that $G^R \leqslant H$, or
  2. there exists a left option H^L such that $G \leqslant H^L$, or
  3. $G = [a]$ and $H = [b]$ are atomic and $a \leqslant b$.

Intuitively, $G \leqslant H$ means that the game H is at least as good for Left as the game G. The following transitivity properties hold for games G, H, K over A: If $G \leqslant H \leqslant K$ then $G \leqslant K$; if $G \lhd H \leqslant K$ then $G \lhd K$; and if $G \leqslant H \lhd K$ then $G \lhd K$. When $G \leqslant H$ and $H \leqslant G$, we say that G and H are *equivalent*. The *value* of a game is its equivalence class; in particular, we say that G and H have the same value if they are equivalent.

A game G is called *locally monotone* if all its left options satisfy $G \leqslant G^L$ and all its right options satisfy $G^R \leqslant G$, and *monotone* if all positions occurring in G are locally monotone. For $n \geqslant 0$, let L_n denote the linearly ordered set with n elements. It was shown in [4] that for $n \leqslant 4$, there exist only finitely many monotone games over L_n up to equivalence. It seems natural to conjecture that this remains true for $n \geqslant 5$, but we will show below that this is not the case: when $n \geqslant 5$, there exist infinitely many non-equivalent monotone games over L_n.

3 An infinite sequence of games over L_5

Let $L_5 = \{-3, -2, -1, 0, 1\}$ be the 5-element linearly ordered set, with its natural order $-3 < -2 < -1 < 0 < 1$. We define the following games and operations on games over L_5:

$$\begin{array}{rcl} \star & = & \{-1 \mid -3\}, \\ M(G) & = & \{1 \mid G\}, \\ P(G) & = & \{G \mid -2\}, \\ P^*(G) & = & \{G \mid \star\}. \end{array}$$

Moreover, for $n \in \mathbb{N}$, we write

$$P^n(G) = \begin{cases} P(G) & \text{when } n \text{ is odd,} \\ P^*(G) & \text{when } n \text{ is even.} \end{cases}$$

Then we define the following sequence of games:

$$\begin{array}{rcl} G_0 & = & 0, \\ G_{n+1} & = & M(P^n(G_n)). \end{array}$$

For example:

$$\begin{array}{rcl} G_0 & = & 0, \\ G_1 & = & M(P^*(0)), \\ G_2 & = & M(P(M(P^*(0)))), \\ G_3 & = & M(P^*(M(P(M(P^*(0)))))). \end{array}$$

Lemma 3.1. *For all n, G_n is monotone.*

Proof. It is easy to see from their definition that the games in the sequence $G_0, G_1, G_2, \dots$ all have the property that the final score is equal to the number of moves made by Left minus the number of moves made by Right. Such games are automatically monotone. To see why, consider the slightly more general class of games G with the property that the final score is equal to the number of moves made by Left minus the number of moves made by Right plus some constant $C \in \mathbb{Z}$. We write $\mathrm{m}(G) = C$ (the *mean value* of G, see [2]). It is then easy to prove by induction that for all such games, $\mathrm{m}(G) \leqslant \mathrm{m}(H)$ implies $G \lhd H$ and $\mathrm{m}(G) < \mathrm{m}(H)$ implies $G \leqslant H$. In particular, since $\mathrm{m}(G^L) = \mathrm{m}(G) + 1$, we have $G \leqslant G^L$, and similarly $G^R \leqslant G$, proving that G is monotone. □

Lemma 3.2. *For all n, $G_n \leqslant G_{n+1}$.*

Proof. To show $G_n \leqslant G_{n+1}$, we must show three things: First, we must show that all left options G_n^L satisfy $G_n^L \lhd G_{n+1}$. When $n = 0$, there is no such left option, and when $n > 0$, the unique left option of G_n is 1. But 1 is also a left option of G_{n+1}, so $1 \lhd G_{n+1}$ as claimed. Second, we must show that all right options G_{n+1}^R satisfy $G_n \lhd G_{n+1}^R$. But the unique right option of G_{n+1} is $P^n(G_n)$, and $G_n \lhd P^n(G_n)$ holds because G_n is a left option of $P^n(G_n)$. Third, we must show that if G_n or G_{n+1} is atomic, then $G_n \lhd G_{n+1}$. But this only happens when $n = 0$. In this case, we must show $0 \lhd G_1$. But this holds because 1 is a left option of G_1 and $0 \leqslant 1$. □

Remark: the proof is not by induction.

Corollary 3.3. *For all n, $0 \leqslant G_n$.*

Proof. From Lemma 3.2 and transitivity, since $0 = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_n$. □

Lemma 3.4. *For all n, $G_{n+1} \nleqslant G_n$.*

Proof. The following claims hold for all n. We prove them by simultaneous complete induction on n. The lemma is claim (8).

(1) $G_n \ntriangleleft -1$.

This follows from Corollary 3.3. Indeed, if $G_n \lhd -1$ were true, then transitivity would imply $0 \lhd -1$, which is absurd.

(2) $P^n(G_n) \nleqslant -1$.

This follows from (1), because G_n is a left option of $P^n(G_n)$.

(3) If n is even, $P^n(G_n) \ntriangleleft -2$.

Since -2 is atomic and $P^n(G_n)$ is not, the only way $P^n(G_n) \lhd -2$ can hold is if some right option H of $P^n(G_n)$ satisfies $H \leqslant -2$. But this is impossible because $\star$ is the unique right option of $P^n(G_n)$ and $\star \nleqslant -2$.

(4) If n is odd, $P^n(G_n) \ntriangleleft \star$.

Since neither $P^n(G_n)$ nor $\star$ is atomic, there are only two ways in which $P^n(G_n) \lhd \star$ could hold. Either some right option H of $P^n(G_n)$ satisfies $H \leqslant \star$; but this is impossible because -2 is the unique right option of $P^n(G_n)$ and $-2 \nleqslant \star$. Or else some left option K of $\star$ satisfies $P^n(G_n) \leqslant K$; but this is impossible by (2) because -1 is the unique left option of $\star$.

(5) If $n > 0$, then $P^n(G_n) \nleqslant P^{n-1}(G_{n-1})$.

When n is even, this follows from (3) because -2 is a right option of $P^{n-1}(G_{n-1})$.

When n is odd, this follows from (4) because $\star$ is a right option of $P^{n-1}(G_{n-1})$.

(6) If $n > 0$, then $G_{n+1} \not\leqslant G_{n-1}$.

For the sake of obtaining a contradiction, suppose $G_{n+1} \leqslant G_{n-1}$. By Lemma 3.2, we have $G_n \leqslant G_{n+1}$. With transitivity, this implies $G_n \leqslant G_{n-1}$. However, this contradicts (8) of the induction hypothesis.

(7) If $n > 0$, then $G_{n+1} \ntriangleleft P^{n-1}(G_{n-1})$.

Because neither G_{n+1} nor $P^{n-1}(G_{n-1})$ is atomic, there are only two ways in which $G_{n+1} \triangleleft P^{n-1}(G_{n-1})$ could hold. Either some right option H of G_{n+1} satisfies $H \leqslant P^{n-1}(G_{n-1})$; but this is impossible by (5) because $P^n(G_n)$ is the unique right option of G_{n+1}. Or else some left option K of $P^{n-1}(G_{n-1})$ satisfies $G_{n+1} \leqslant K$; but this is impossible by (6) because G_{n-1} is the unique left option of $P^{n-1}(G_{n-1})$.

(8) $G_{n+1} \not\leqslant G_n$.

When $n = 0$, this holds by direct calculation: $G_1 \not\leqslant 0$ because 1 is a left option of G_1 and $1 \ntriangleleft 0$.

When $n > 0$, this follows from (7) because $P^{n-1}(G_{n-1})$ is a right option of G_n. □

Corollary 3.5. *There are infinitely many non-equivalent monotone games over the* 5*-element linearly ordered atom poset* L_5.

Proof. By Lemmas 3.2 and 3.4, we have $G_n < G_{n+1}$ for all n. In particular, the sequence $G_0, G_1, \ldots$ consists of infinitely many non-equivalent games. Moreover, they are monotone by Lemma 3.1. □

Corollary 3.5 completes the classification of atom posets into whether there exist finitely or infinitely many game values. Specifically, it provides the last remaining piece of the following theorem:

Theorem 3.6. *Let A be a poset. Then the class of monotone game values over A is finite if A is a linear order of 4 or fewer elements. In all other cases, there are infinitely many monotone values.*

Proof. It was shown in [4, Prop. 10.2] that there are infinitely many monotone game values over A when A has two incomparable elements. This takes care of all cases where A is not linearly ordered. It was further shown in [4, Sec. 10.2] that there are only finitely many monotone game values over A when A is a linearly ordered set of size 1, 2, 3, or 4. (In this case, there are 1, 3, 8, or 31 such values, respectively). The case of the zero-element poset is trivial, as there are no game values at all. The only remaining cases are linearly ordered sets of 5 or more elements (including infinite ones). In these cases, there are infinitely many monotone game values by Corollary 3.5. Note that if the atom poset has strictly more than 5 elements, one may simply disregard the additional atoms. □

We conclude this paper with a remark that may shed some light on the properties of the games G_n.

Remark 3.7. As mentioned in the proof of Lemma 3.1, the games in the sequence $G_0, G_1, G_2, \ldots$ all have the property that the final score is equal to the number of moves made by Left minus the number of moves made by Right. In combinatorial game terminology, these games have *constant temperature 1*, because each move shifts the average outcome by exactly 1 in the direction that favors the player who made the move. As we already saw, such games are automatically monotone. Moreover, such games are equivalent to normal-play games in the following sense: if Left goes second in G and the players alternate, the outcome will be 0 if and only if Left gets the last move, and -1 otherwise. Let $\mathrm{np}(G)$ be the normal-play game obtained from G by replacing every atom by $0 = \{\ |\ \}$. Then $0 \leqslant G$ if and only if Left has a second-player strategy guaranteeing outcome 0, if and only if Left has a second-player strategy guaranteeing the last move, if and only if Left has a second-player winning strategy in $\mathrm{np}(G)$, if and only if $0 \leqslant \mathrm{np}(G)$. Moreover, the same observation holds for comparison games as well, so that $G \leqslant H$ if and only if $\mathrm{np}(G) \leqslant \mathrm{np}(H)$. We can therefore see that the strictly increasing sequence of monotone games $G_0 < G_1 < \ldots$ corresponds to a strictly increasing sequence $\mathrm{np}(G_0) < \mathrm{np}(G_1) < \ldots$ of (rather specially constructed) normal-play games.

References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning Ways for Your Mathematical Plays.* A. K. Peters, 2nd edition, 2001.

[2] J. H. Conway. *On Numbers and Games.* A. K. Peters, 2nd edition, 2001.

[3] E. Demer, P. Selinger, and K. Wang. All passable games are realizable as monotone set coloring games. Available from arXiv:2111.10351, Nov. 2021.

[4] P. Selinger. On the combinatorial value of Hex positions. Available from arXiv:2101.06694, Jan. 2021.