Order-Incompleteness and Finite Lambda Models
Extended Abstract

Peter Selinger*
Department of Mathematics
University of Pennsylvania
209 S. 33rd Street
Philadelphia, PA 19104-6395
selinger@math.upenn.edu

Abstract

Many familiar models of the type-free lambda calculus are constructed by order theoretic methods. This paper provides some basic new facts about ordered models of the lambda calculus. We show that in any partially ordered model that is complete for the theory of β- or $\beta\eta$-conversion, the partial order is trivial on term denotations. Equivalently, the open and closed term algebras of the type-free lambda calculus cannot be non-trivially partially ordered. Our second result is a syntactical characterization, in terms of so-called generalized Mal'cev operators, of those lambda theories which cannot be induced by any non-trivially partially ordered model. We also consider a notion of finite models for the type-free lambda calculus. We introduce partial syntactical lambda models, which are derived from Plotkin's syntactical models of reduction, and we investigate how these models can be used as practical tools for giving finitary proofs of term inequalities. We give a 3-element model as an example.

1 Introduction

Perhaps the most important contribution in the area of mathematical programming semantics was the discovery, by D. Scott in the late 1960's, that models for the type-free lambda calculus could be obtained by a combination of order-theoretic and topological methods. A long tradition of research in domain theory ensued, and Scott's methods have been successfully applied to many aspects of programming semantics.

In this paper we establish some basic new facts about ordered models of the type-free lambda calculus. We show that the standard open and closed term algebras are *unorderable*, *i.e.* they cannot be non-trivially partially ordered as combinatory algebras. We also give a syntactic characterization, in terms of Mal'cev operators, of those combinatory algebras which are *absolutely unorderable* in the sense that they cannot be embedded in any orderable algebra.

In the second part of the paper (Section 4), we develop a notion of finite models for the type-free lambda calculus. It has long been known that a model of the lambda calculus in the traditional sense can never be finite or even recursive [1]. We circumvent this problem by introducing *partial syntactical lambda models*, which differ from the usual syntactical lambda models [1] in that their operations are only partially defined, hence giving denotations only to a subset of lambda terms. Such models can be finite, and consequently term denotations are effectively computable. We give an example of a non-trivial 3-element model.

Let us briefly present the main issues discussed in this paper.

Order-Incompleteness. Many familiar models of the type-free lambda calculus, like Scott's D_∞ and $P\omega$, are constructed by order theoretic methods. It is a natural question to ask whether the class of such ordered models is *complete* with respect to the lambda calculus. More precisely, every model defines, via the interpretation function, a lambda theory, by which we mean a congruence relation on closed lambda terms with or without constants, closed under the β-rule. By $\underline{\lambda\beta}$ we denote the minimal theory, *i.e.* the theory of β-conversion. By $\underline{\lambda\beta\eta}$, we denote the theory of $\beta\eta$-conversion. We consider the following two completeness questions:

(i) Is there a non-trivially partially ordered model whose theory is $\underline{\lambda\beta}$ (or $\underline{\lambda\beta\eta}$)?

(ii) Does *every* lambda theory arise as the theory of a non-trivially partially ordered model?

*This research was supported by a graduate fellowship from the Institute for Research in Cognitive Science and by a dissertation fellowship from the School of Arts and Sciences at the University of Pennsylvania.

Although these questions are of similar nature, (i) is easy, while (ii) is hard. Question (i) asks for a partially ordered model which is *complete* for β- or $\beta\eta$-conversion. Recall that neither D_∞ nor $P\omega$ are complete in this sense, since in these models the meanings of unsolvable terms are equated [6, 13, 1]. In Corollary 8 below, we construct partially ordered models that are β- and $\beta\eta$-complete. Question (i) becomes more difficult, and more interesting, when it is specialized to smaller classes of models. Di Gianantonio, Honsell and Plotkin [2] give a positive answer with respect to models that satisfy a certain weak ω_1-continuity condition. It is an open problem whether (i) still holds when specialized to the class of *topological models*, *i.e.* models which arise as reflexive objects in a cartesian closed category of complete partial orders and Scott-continuous functions. This is the long-standing *topological completeness problem*, see [5].

We will show in Section 2 that all models that satisfy condition (i) share an interesting property: the denotations of lambda terms necessarily form a discrete subset, *i.e.* denotations are pairwise incomparable. This is a consequence of our Theorem 3: the standard open and closed term algebras of the type-free lambda calculus cannot be non-trivially partially ordered as combinatory algebras.

Question (ii) asks for the existence of *order-incomplete* lambda theories. We call a theory order-incomplete if it does not arise as the theory of a non-trivially partially ordered model. In Section 3, we give a syntactical characterization of such theories: the order-incomplete lambda theories are exactly those theories that have a family of so-called *generalized Mal'cev operators*. Mal'cev operators are known from universal algebra [8, 12, 4], and they arise here because of their ability to interfere with orders.

For the smaller class of topological models, (ii) is known to be false: Honsell and Ronchi Della Rocca [5] give a lambda theory that is not induced by any topological model.

Finite Lambda Models. Our notion of finite models for the type-free lambda calculus is derived from a notion of models of reduction. Models of reduction have been considered by different authors [3, 7, 9], and we will focus here on a formulation which is given by Plotkin [9] in the spirit of the familiar *syntactical lambda models* [1]. Informally, by a *model of convertibility* for a lambda calculus, we mean a model with a soundness property of the form

$$M \cong N \Rightarrow [\![M]\!] = [\![N]\!],$$

where $\cong$ is *e.g.* β- or $\beta\eta$-convertibility, and $[\![\ ]\!]$ is a 'meaning' function of lambda terms. On the other hand, a *model of reduction* has a soundness property of the form

$$M \longrightarrow N \Rightarrow [\![M]\!] \leq [\![N]\!],$$

where $\longrightarrow$ is *e.g.* β- or $\beta\eta$-reduction, and $\leq$ is some order on meanings of terms.

It is known that models of convertibility for the type-free lambda calculus are never finite or even recursive [1]. By contrast, models of reduction can quite possibly be finite, and we shall see that they are easy to construct — but they do not in general yield any information about convertibility of terms. Our notion of *partial models* combines features of both.

The key observation that leads to the definition of partial models is that in certain models of reduction, a restricted form of reasoning about convertibility is possible. Specifically, this is the case if the underlying order is flat, *i.e.* of the form $X_\perp$, where X is a set. A partial model can therefore be understood as a flat model of reduction. After giving the relevant definitions in 4.1 and 4.2, we investigate methods of defining such models in practice. Such a method is given in 4.3, and an example is worked out in 4.4 of how a 3-element partial model can be used to establish a non-trivial term inequality.

2 Lambda terms cannot be ordered

The main result of this section is that the open and closed term algebras of the type-free lambda calculus do not allow a non-trivial partial order. It is understood that by an ordered model we mean one where the order is compatible with the model structure. Since we formulate the results in terms of combinatory algebras, this simply means that application is a monotone operation. We follow Barendregt's notation for the lambda calculus [1].

Definition An *applicative structure* $(X, \cdot)$ is a set X with a binary operation. A *combinatory algebra* $(X, \cdot, k, s)$ is an applicative structure with distinguished $k, s \in X$ satisfying $kxy = x$ and $sxyz = xz(yz)$. Recall that every combinatory algebra is *combinatory complete*, *i.e.*, for any polynomial expression $p(x_1, \ldots, x_n)$ there is an element $f \in X$ with $fx_1 \ldots x_n = p(x_1, \ldots, x_n)$.

We say that a preorder $\leq$ on an applicative structure $(X, \cdot)$ is *compatible* if $a \leq b$ implies $f \cdot a \leq f \cdot b$ for all $f, a, b \in X$. Note that by combinatory completeness, this implies $p(a) \leq p(b)$ for *any* polynomial expression $p(x)$; in particular, for a compatible preorder, application is *monotone* in both arguments.

A preorder is said to be *discrete* if $a \leq b$ implies $a = b$, *indiscrete* if $a \leq b$ holds for all a, b, and *trivial* if it is either descrete or indiscrete. By convention, we will refer to preorders that merely satisfy $a \leq b \Rightarrow b \leq a$ as *symmetric*. A symmetric partial order is, of course, discrete and hence trivial.

A combinatory algebra is called *unorderable* if the only compatible partial orders on it are the trivial ones. Such algebras have been previously known to exist. For example, Plotkin [10] has recently exhibited a *finitely separable* algebra, a property which implies unorderability. Here $(X, \cdot)$ is said to be finitely separable if for every finite subset $A \subseteq X$, every function $f\colon A \to X$ is the restriction of some $\hat{f} \in X$, meaning that for all $a \in A$, $f(a) = \hat{f} \cdot a$. Finitely separable combinatory algebras do not allow non-trivial preorders, for if $a < b$ for some $a, b \in X$, then $x \leq y$ for all $x, y \in X$ via some $\hat{f} \in X$ with $\hat{f} \cdot a = x$ and $\hat{f} \cdot b = y$.

The present result differs from this, because our unorderable algebras, the open and closed term algebras of the type-free lambda calculus, occur "naturally".

Definition Let Λ_C be the set of (α-equivalence classes of) untyped lambda terms, built from constants in C and some countable supply $\mathcal{V}$ of variables. Similarly, Λ_C^0 is the set of closed terms. For $\Lambda_\emptyset$ we simply write Λ.

The *open term algebra* of the $\lambda\beta$-calculus is the combinatory algebra $(\Lambda_C/{=_\beta}, \bullet, K, S)$, where $\bullet$ is the application operation on terms, and K and S are the terms $\lambda xy.x$ and $\lambda xyz.xz(yz)$, respectively. The *closed term algebra* $(\Lambda_C^0/{=_\beta}, \bullet, K, S)$ is defined analogously, and similarly for the $\lambda\beta\eta$-calculus.

Note that these term algebras are not finitely separable: *e.g.* the terms $\omega = (\lambda x.xx)(\lambda x.xx)$ and $I = \lambda x.x$ cannot be separated, since the first one is unsolvable. Also, the term algebras allow non-trivial *pre*orders: for instance, two terms are ordered if and only if their meanings in the D_∞ model are ordered.

We start with a lemma, to be proved in 4.4 below:

Lemma 1 *There is a closed term A of the type-free lambda calculus with $Auuut =_\beta Auttt$, but $Auuut \neq_{\beta\eta} Auutt \neq_{\beta\eta} Auttt$ for variables $u \neq t$.* □

The lemma is relevant to orders because of the following observation: if $\leq$ is any compatible partial order on open terms and $u \neq t$ are variables, then $u \not\leq t$. For suppose $u \leq t$, then $Auuut \leq Auutt \leq Auttt = Auuut$, hence $Auuut = Auutt$ by antisymmetry, which contradicts Lemma 1.

This shows that any compatible partial order on the open term algebra is trivial on variables. To show that it is trivial on arbitrary terms $\mathbf{u}$, $\mathbf{t}$, we need the following technical fact: If $\mathbf{u}$, $\mathbf{t}$ are terms and s is a fresh variable, then any inequality which holds for variables u and t will also hold for $s\mathbf{u}$ and $s\mathbf{t}$. In this sense, we can say that $s\mathbf{u}$ and $s\mathbf{t}$ behave like *generic arguments*. This is summarized in Lemma 2. Here $=_{\beta(\eta)}$ denotes either $=_\beta$ or $=_{\beta\eta}$.

Lemma 2 *Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be terms such that $\mathbf{u}_i \neq_{\beta(\eta)} \mathbf{u}_j$ for $i \neq j$. If s is a variable not free in $\mathbf{u}_1, \ldots, \mathbf{u}_n$, then $s\mathbf{u}_1, s\mathbf{u}_2, \ldots, s\mathbf{u}_n$ behave like generic arguments, i.e. for all terms M, N with $s \notin \mathrm{FV}(M, N)$,*

$$M(s\mathbf{u}_1)(s\mathbf{u}_2)\ldots(s\mathbf{u}_n) =_{\beta(\eta)} N(s\mathbf{u}_1)(s\mathbf{u}_2)\ldots(s\mathbf{u}_n)$$

implies

$$Mx_1x_2\ldots x_n =_{\beta(\eta)} Nx_1x_2\ldots x_n.$$

Proof Idea: An easy syntactic proof is possible by observing that the subterms of the form $(s\mathbf{u})$ never disappear under $\beta\eta$-reductions. A more semantic proof can be given if one uses Plotkin's separability result [10] to embed the open term algebra in a separable algebra. There, one can choose s to map $\mathbf{u}_1, \ldots, \mathbf{u}_n$ to $x_1, \ldots, x_n$, respectively. □

Theorem 3 *Let $\mathcal{M}$ be the open or the closed term algebra of the $\lambda\beta$- or $\lambda\beta\eta$-calculus. Then $\mathcal{M}$ does not allow a non-trivial compatible partial order.*

Proof: Equality in $\mathcal{M}$ is denoted by $=_{\beta(\eta)}$. Take a compatible partial order $\leq$. Let $\mathbf{u}, \mathbf{t}$ be terms with $\mathbf{u} \leq \mathbf{t}$. Let A be as in Lemma 1, and let s be a fresh variable. Then by compatibility,

$$\begin{array}{rcl} \lambda s.A(s\mathbf{u})(s\mathbf{u})(s\mathbf{u})(s\mathbf{t}) & \leq & \lambda s.A(s\mathbf{u})(s\mathbf{u})(s\mathbf{t})(s\mathbf{t}) \\ & \leq & \lambda s.A(s\mathbf{u})(s\mathbf{t})(s\mathbf{t})(s\mathbf{t}) \\ & =_{\beta(\eta)} & \lambda s.A(s\mathbf{u})(s\mathbf{u})(s\mathbf{u})(s\mathbf{t}), \end{array}$$

hence, by antisymmetry,

$$A(s\mathbf{u})(s\mathbf{u})(s\mathbf{u})(s\mathbf{t}) =_{\beta(\eta)} A(s\mathbf{u})(s\mathbf{u})(s\mathbf{t})(s\mathbf{t})$$

Applying Lemma 2 to $M = \lambda ut.Auuut$ and $N = \lambda ut.Auutt$, one gets $\mathbf{u} =_{\beta(\eta)} \mathbf{t}$. Consequently, the order is trivial. □

Corollary 4 *In any partially ordered model of the type-free lambda calculus which is complete for one of the theories $\underline{\lambda\beta}$ or $\underline{\lambda\beta\eta}$, the denotations of terms are discretely ordered.* □

3 A characterization of absolutely unorderable algebras

We have shown that the combinatory algebra of open lambda terms cannot be non-trivially ordered. Nevertheless, it follows from Di Gianantonio, Honsell and Plotkin [2] that it can be *embedded* in an orderable algebra. It is an interesting question whether there is a combinatory algebra which is *absolutely unorderable* in the sense that it cannot even be embedded in an orderable one. Plotkin conjectures in [10] that the answer is yes.

The problem of absolute unorderability is closely related to the completeness question (ii) in the introduction. Clearly, the lambda theories that cannot be realized in a non-trivially partially ordered model are those whose term algebra is absolutely unorderable.

We will now characterize absolutely unorderable algebras as those algebras that have a family of *generalized Mal'cev operators*. Such operators are known from the study of Mal'cev varieties in universal algebra. They are relevant here because of the way they interfere with orders.

3.1 Absolutely unorderable algebras and generalized Mal'cev operators

A combinatory algebra $\mathbf{A}$ is *absolutely unorderable* if there is no embedding of combinatory algebras $\mathbf{A} \to \mathbf{B}$ such that $\mathbf{B}$ allows a non-trivial compatible partial order. The following theorem characterizes such algebras. As usual, $\mathbf{A}[u,t]$ denotes the combinatory algebra obtained by freely adjoining indeterminates u, t to $\mathbf{A}$.

Theorem 5 *The following are equivalent:*

1. $\mathbf{A}$ is absolutely unorderable.
2. For every compatible preorder $\leq$ on $\mathbf{A}[u,t]$, if $u \leq t$ then $t \leq u$.
3. For some $n \geq 1$, there exist elements $\mathbf{M}_1, \ldots, \mathbf{M}_n \in \mathbf{A}$, called *generalized Mal'cev operators*, such that, for indeterminates u, t,

$$\begin{array}{rcll} t & = & \mathbf{M}_1 tuu & \\ \mathbf{M}_1 ttu & = & \mathbf{M}_2 tuu & \\ \mathbf{M}_2 ttu & = & \mathbf{M}_3 tuu & \qquad (\textit{Malcev}_n) \\ & \vdots & & \\ \mathbf{M}_n ttu & = & u & \qquad \square \end{array}$$

Remark: We say an equation $p(u,t) = q(u,t)$ holds *for indeterminates* u, t if it holds in $\mathbf{A}[u,t]$. This is a stronger condition than being *satisfied* in $\mathbf{A}$, which is usually denoted by $\mathbf{A} \models p(u,t) = q(u,t)$, and which means that the equation holds for all $u, t \in \mathbf{A}$. If $\mathbf{A}$ is extensional, or more generally if $\mathbf{A}$ is a *lambda model* [1], the two concepts coincide.

In the case $n = 1$, the equations $(\textit{Malcev}_1)$ have the simple form $t = \mathbf{M}tuu$ and $\mathbf{M}ttu = u$. This is the usual Mal'cev operator known from universal algebra. Generalized Mal'cev operators for $n \geq 1$ have been used by Hagemann and Mitschke [4] to characterize n-permutable classes of universal algebras.

It is worth noting that Theorem 5 holds not just for combinatory algebras, but in fact in any equational variety. Specifically:

Theorem 6 *Let* $\mathbf{A}$ *be an algebra in an equational variety of algebras. Then* $\mathbf{A}$ *is absolutely unorderable if and only if, for some* $n \geq 1$*, there are* $M_1(x,y,z), \ldots, M_n(x,y,z) \in \mathbf{A}[x,y,z]$*, such that the equations* $(\textit{Malcev}_n)$ *are satisfied in* $\mathbf{A}[u,t]$. $\square$

3.2 Generalized Mal'cev operators and theories

We call a lambda theory *order-incomplete* if it does not arise as the theory of any non-trivially partially ordered lambda algebra, *i.e.* if no ordered model is complete for it. In the introduction, we were asking in question (ii) whether an order-incomplete theory exists. Applying Theorem 5, and using the duality between models and theories, we can now also characterize order-incomplete theories in terms of generalized Mal'cev operators. Lambda algebras are defined in [1].

Theorem 7 *For a lambda theory* $\mathcal{T}$*, the following are equivalent:*

1. $\mathcal{T}$ is order-incomplete.
2. The closed term algebra $\Lambda^0_C/\mathcal{T}$ is absolutely unorderable.
3. The open term algebra $\Lambda_C/\mathcal{T}$ is absolutely unorderable.
4. $\mathcal{T}$ has a family of generalized Mal'cev operators, *i.e.* there are closed terms $\mathbf{M}_1, \ldots, \mathbf{M}_n \in \Lambda^0_C$ such that $\mathcal{T} \vdash t = \mathbf{M}_1 tuu$ etc. for variables u, t.
5. There is no interpretation of $\mathcal{T}$ in any non-symmetrically preordered lambda algebra. $\square$

Note that if $\mathbf{T}_2$ is an extension of an order-incomplete theory $\mathbf{T}_1$, then by 4., $\mathbf{T}_2$ is also order-incomplete. Also note that because of 5., an order-incomplete theory does not have *any* (not necessarily complete) non-trivially ordered models. In particular, neither $\underline{\lambda\beta}$ nor $\underline{\lambda\beta\eta}$ are order-incomplete theories. Thus we can positively answer question (i) from the introduction:

Corollary 8 *Both* $\underline{\lambda\beta}$ *and* $\underline{\lambda\beta\eta}$ *arise as the theory of a non-trivially partially ordered model.* $\square$

3.3 Mal'cev operators and consistency

At the beginning of this section, we asked whether an absolutely unorderable combinatory algebra exists. In light of Theorem 5, this is the case if and only if, for some n, the equations $(\textit{Malcev}_n)$ are consistent with combinatory logic. Unfortunately, this is not known, except in the cases $n = 1$ and $n = 2$.

Let Y be any fixpoint operator of combinatory logic, and write $\mu x.M$ for $Y(\lambda x.M)$. The operator μ satisfies the fixpoint property:

$$\mu x.A(x) = A(\mu x.A(x)). \qquad (\textit{fix})$$

The *diagonal axiom* is

$$\mu y.\mu x.A(x,y) = \mu x.A(x,x). \qquad (\Delta)$$

Theorem 9 (Plotkin, A. Simpson) *Assuming the diagonal axiom,* (*Malcev*$_n$) *is inconsistent with combinatory logic for all* n.
Proof: Let x be arbitrary. Let $A = \mu z.\mathbf{M}_1 xzz$. Then $A = \mu z.x = x$. Also,

$$\begin{array}{rcll} x = A & = & \mu z.\mathbf{M}_1 xzz & \\ & = & \mu y.\mu z.\mathbf{M}_1 xyz & \text{by } (\Delta) \\ & = & \mu z.\mathbf{M}_1 xxz & \text{by } (\mathit{fix}) \\ & = & \mu z.\mathbf{M}_2 xzz & \text{by } (\mathit{Malcev}_n) \\ & = & \ldots & \\ & = & \mu z.\mathbf{M}_{n-1} xxz & \\ & = & \mu z.z & \text{by } (\mathit{Malcev}_n) \end{array}$$

Hence $x = \mu z.z$ for all x, which is an inconsistency. □

Theorem 10 (Plotkin, Simpson) (*Malcev*$_1$) *is inconsistent with combinatory logic.*
Proof: Suppose $\mathbf{M}$ is a Mal'cev operator. Let x be arbitrary and let $A = \mu y.\mu z.\mathbf{M}xyz$. Then

$$A \overset{(\mathit{fix})}{=} \mu z.\mathbf{M}xAz \overset{(\mathit{fix})}{=} \mathbf{M}xAA \overset{(\mathit{Malcev}_1)}{=} x,$$

hence $x = \mu z.\mathbf{M}xAz = \mu z.\mathbf{M}xxz = \mu z.z$. □

Theorem 11 (Plotkin, Selinger) (*Malcev*$_2$) *is inconsistent with combinatory logic.*
Proof: Suppose $\mathbf{M}_1$ and $\mathbf{M}_2$ are operators satisfying (*Malcev*$_2$). Define A and B by mutual recursion such that

$$\begin{array}{rcl} A & = & \mu x.f(\mathbf{M}_1 xAB)(\mathbf{M}_1 xAB) \\ B & = & \mu y.\mu z.f(\mathbf{M}_2 ABy)(\mathbf{M}_2 ABz). \end{array}$$

Then

$$\begin{array}{rcll} B & = & f(\mathbf{M}_2 ABB)(\mathbf{M}_2 ABB) & \text{by } (\mathit{fix}) \\ & = & f(\mathbf{M}_1 AAB)(\mathbf{M}_1 AAB) & \text{by } (\mathit{Malcev}_2) \\ & = & A. & \text{by } (\mathit{fix}) \end{array}$$

So $\mu x.fxx = \mu x.f(\mathbf{M}_1 xAA)(\mathbf{M}_1 xAA) = A = B = \mu y.\mu z.f(\mathbf{M}_2 AAy)(\mathbf{M}_2 AAz) = \mu y.\mu z.fyz$, which is the diagonal axiom. By Theorem 9, this leads to an inconsistency. □

4 Finite models for the lambda calculus

4.1 Models of reduction

Definition 12 An *ordered applicative structure* $(P, \cdot)$ is a poset P with a monotone binary operation $\cdot: P \times P \to P$. Let $P^{\mathcal{V}}$ be the set of all *valuations*, *i.e.* functions from variables to P. A *syntactical model of* β*-reduction* $(P, \cdot, [\![\ ]\!])$ [9] is an ordered applicative structure together with a *meaning function*

$$[\![\ \cdot\ ]\!].: \Lambda \times P^{\mathcal{V}} \to P$$

such that the following hold:

1. $[\![x]\!]_\rho = \rho(x)$
2. $[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho$
3. $[\![\lambda x.M]\!]_\rho \cdot a \le [\![M]\!]_{\rho(x:=a)}$, for all $a \in X$
4. $\rho|_{\mathrm{FV}(M)} = \rho'|_{\mathrm{FV}(M)} \Rightarrow [\![M]\!]_\rho = [\![M]\!]_{\rho'}$
5. $(\forall a.[\![M]\!]_{\rho(x:=a)} \le [\![N]\!]_{\rho(x:=a)}) \Rightarrow [\![\lambda x.M]\!]_\rho \le [\![\lambda x.N]\!]_\rho$

Moreover, we say $(P, \cdot, [\![\ ]\!])$ is a *syntactical model of* $\beta\eta$*-reduction*, if it satisfies the additional property

6. $[\![\lambda x.Mx]\!]_\rho \le [\![M]\!]_\rho$, if $x \notin \mathrm{FV}(M)$.

Note that if P is a *discrete* partial order, we obtain the usual syntactical lambda models (of convertibility) [1].

Also note that 1.–3. do not form an inductive definition; rather they state properties of a function $[\![\ ]\!]$ which is given *a priori*. In particular, 3. does not uniquely determine the meaning of an abstraction.

Unlike in ordered models of convertibility, where the order relation is to be understood as 'information order', here we are dealing with a *reduction order*: $a \le b$ intuitively means a reduces to b. More precisely, the following properties hold:

Proposition 13 (Plotkin [9]) *The following are properties of models of* β*-reduction:*

1. **Monotonicity.** If $\rho(x) \le \rho'(x)$ for all x, then $[\![M]\!]_\rho \le [\![M]\!]_{\rho'}$.
2. **Substitution.** $[\![M[N/x]]\!]_\rho = [\![M]\!]_{\rho(x:=[\![N]\!]_\rho)}$.
3. **Soundness for reduction.** If $M \xrightarrow{\beta} N$, then $[\![M]\!]_\rho \le [\![N]\!]_\rho$. In a model of $\beta\eta$-reduction: If $M \xrightarrow{\beta\eta} N$, then $[\![M]\!]_\rho \le [\![N]\!]_\rho$. □

Definition 14 A *categorical model of* β*-reduction* (P, p, e) is given by an object P in an order-enriched cartesian closed category, together with a pair of morphisms $p: P \to P^P$ and $e: P^P \to P$, such that $p \circ e \le \mathrm{id}_{P^P}$. If moreover $e \circ p \le \mathrm{id}_P$, then (P, p, e) is a *categorical model of* $\beta\eta$*-reduction*.

Such categorical models of reduction have been studied by various authors, *e.g.* by Girard [3] for the case of qualitative domains, or by Jacobs *et al.* [7], where they are called models of expansion. For a detailed discussion of these and other sources, see Plotkin [9].

There is an evident way of defining a syntactical model of β-reduction $(P, \cdot, [\![\ ]\!])$ from a concrete categorical model of β-reduction, by letting $a \cdot b := p(a)(b)$ and $[\![\lambda x.M]\!]_\rho := e(\lambda a.[\![M]\!]_{\rho(x:=a)})$. Similarly for $\beta\eta$. This is completely analogous to Scott's familiar rendition of a model of convertibility as a reflexive object in a cartesian closed category [11].

Constructing models of reduction is much easier than the corresponding task of finding models of convertibility. Indeed, for a given $p\colon P \to P^P$, there are a few obvious choices for e: a minimal choice is always possible if P has a least element $\bot$. In this case, let e be the constant $\bot$ function, which amounts to declaring $[\![\lambda x.M]\!]_\rho$ to be always undefined. On the other hand, if p has a right adjoint e, then e is maximal with $p \circ e \leq \mathrm{id}$. In this case, $[\![\lambda x.M]\!]_\rho$ is the maximal $b \in P$ such that for all $a \in P$, $b \cdot a \leq [\![M]\!]_{\rho(x:=a)}$.

4.2 Partial Models

We have already noted that the soundness property for models of reduction does not in general yield any information about convertibility. The best statement one can make is that, if the reduction under consideration is confluent (Church-Rosser), then $[\![M]\!]_\rho$ and $[\![N]\!]_\rho$ must be *compatible*, *i.e.*

$$M \cong N \Rightarrow \exists c \in P.[\![M]\!]_\rho \leq c \text{ and } [\![N]\!]_\rho \leq c.$$

For this reason we will be especially interested in partial orders P that have lots of incompatible pairs. An important special case arises with $(P, \cdot, [\![\ ]\!])$ where $P = X_\bot$ is a *flat* order, and where $\cdot\colon P \times P \to P$ is *strict* in each argument. In this case, it is convenient to consider $\bot$ as the *undefined* element, and $\cdot$, $[\![\ ]\!]$ as *partial* functions. This gives rise to the following definition.

Definition 15 A *partial applicative structure* $(X, \cdot)$ is a set X with a partial binary operation $\cdot\colon X \times X \rightharpoonup X$. Let $\mathrm{Val}(X)$ be the set of partial valuations $\mathcal{V} \rightharpoonup X$. A *partial syntactical lambda model* $(X, \cdot, [\![\ ]\!])$, or *partial model* for short, is given by a partial applicative structure together with a partial map

$$[\![\cdot]\!].\colon \Lambda_X \times \mathrm{Val}(X) \rightharpoonup X,$$

such that properties 1.–5. in Definition 12 hold, where $=$ is replaced by Kleene equality and $\leq$ by directed equality. Moreover, if property 6. holds, $(X, \cdot, [\![\ ]\!])$ is a *partial $\beta\eta$-model*.

Kleene equality is defined as follows: $A = B$ if and only if A and B are either both undefined or both defined and equal. Directed equality, which we often denote by $A \succeq\!\!= B$, means that if A is defined, then so is B and they are equal.

The idea of using partiality in models for the lambda calculus is not new. In fact, Kleene's "first model", which consists of Gödel numbers of partial recursive functions and their application, is partial. Partial models in our sense, however, need not even be partial combinatory algebras. In particular, here we do not even assume that the meanings of the combinators S and K are necessarily defined.

In any particular partial syntactical lambda model, the class of terms which denote in it is not given *a priori*, but derived. The following soundness properties ensure that this class is closed under reduction. Note that for partial models, we have two notions of soundness: the one for reduction like before, and now, as a trivial consequence, one for convertibility.

Proposition 16 *The following are properties of partial models:*

1. **Soundness for reduction.** If $M \xrightarrow{\beta} N$, then $[\![M]\!]_\rho \succeq\!\!= [\![N]\!]_\rho$.
2. **Soundness for convertibility.** If $M =_\beta N$, and if $[\![M]\!]_\rho$ and $[\![N]\!]_\rho$ are both defined, then $[\![M]\!]_\rho = [\![N]\!]_\rho$.
3. In a partial $\beta\eta$-model, the respective properties hold for $=_{\beta\eta}$. □

4.3 Methods of construction

For a given partial model, the semantic interpretation function $[\![\ ]\!]$ is given *a priori*, satisfying certain properties. For all practical purposes, one desires to be able to define $[\![\ ]\!]$ inductively. This is in general not possible. It turns out, however, that if $(X, \cdot)$ is *strongly extensional*, then $[\![\ ]\!]$ can be defined inductively in a maximal way:

Definition 17 A partial applicative structure $(X, \cdot)$ is *weakly extensional* if for all elements $a, b \in X$, whenever $\forall x \in X.ax \succeq\!\!= bx$ then $a = b$. It is *strongly extensional* if for all elements $a, b \in X$, whenever ($\forall x \in X.ax$ and bx defined $\Rightarrow$ $ax = bx$), then $a = b$.

Note that strong extensionality implies weak extensionality, and that in the total case, both coincide with extensionality.

For a given partial applicative structure $(X, \cdot)$, let $P = X_\bot$ be the corresponding flat order, and define $p\colon P \to P^P$ by $p(a)(b) = a \cdot b$. The following proposition relates weak extensionality to the η-rule, and strong extensionality to the existence of a right adjoint to p:

Proposition 18 *If $(X, \cdot)$ is weakly extensional and $|X| \geq 2$, then every partial model $(X, \cdot, [\![\ ]\!])$ is a partial $\beta\eta$-model. If $(X, \cdot)$ is strongly extensional, then $p\colon P \to P^P$ has a right adjoint.* □

Hence, if we start with a strongly extensional applicative structure $(X, \cdot)$, it is possible to define $[\![\ ]\!]$ inductively, using the adjointness $p \dashv e$ like in the discussion following Definition 14. The resulting definition of $[\![\ ]\!]$ is maximal, and it is summarized in the following corollary:

Corollary 19 *If $(X, \cdot)$ is a strongly extensional applicative structure, then the following defines a partial $\beta\eta$-model:*

1. $[\![x]\!]_\rho = \rho(x)$

2. $[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho$

3. $[\![\lambda x.M]\!]_\rho$ is defined iff there exists $b \in X$ with $b \cdot a \succeq [\![M]\!]_{\rho(x:=a)}$ for all $a \in X$. By strong extensionality, such an element b is necessarily unique. Define $[\![\lambda x.M]\!]_\rho = b$.

Moreover, if $[\![\ ]\!]$ is defined in this way, then for all $n \geq 1$ and $b \in X$, one has $b = [\![\lambda x_1 \dots x_n.P]\!]_\rho$ if and only if for all $a_1 \dots a_n \in X$, $b \cdot a_1 \cdots a_n \succeq [\![P]\!]_{\rho(x_1:=a_1)\dots(x_n:=a_n)}$. In particular, if such b exists, it is unique; otherwise $[\![\lambda x_1 \dots x_n.P]\!]_\rho$ is undefined.
Proof: The last claim follows by induction on n. □

4.4 Example: A 3-element partial model

We demonstrate the methods of the previous subsection by supplying a proof of Lemma 1 above. The proof illustrates how a simple partial model can be 'put to action'.

Lemma *There is a closed term A of the type-free lambda calculus with $Auuut =_\beta Auttt$, but $Auuut \neq_{\beta\eta} Auutt \neq_{\beta\eta} Auttt$ for variables $u \neq t$.*
Proof: Define terms

$$\begin{aligned} h &= \lambda zyx.zzy(zzy(zzyx)) \\ f &= hh \\ A &= \lambda uvwt.\lambda x.fu(fv(fw(ftx))). \end{aligned}$$

Then for all x, y:

$$fyx \xrightarrow{\beta} fy(fy(fyx)),$$

hence for all u, t:

$$\begin{aligned} \lambda x.fu(ftx) &\xrightarrow{\beta} \lambda x.fu(fu(fu(ftx))) = Auuut \\ \lambda x.fu(ftx) &\xrightarrow{\beta} \lambda x.fu(ft(ft(ftx))) = Auttt. \end{aligned}$$

To see that $Auuut \neq_{\beta\eta} Auutt$ for variables u and t, we will construct a partial model with as little as 3 elements. Let $X = \{\mathrm{k}, 0, 1\}$, and let $\cdot$ be defined by the following 'multiplication table':

$\cdot$	k	0	1
k	0	0	0
0	0	0	1
1	0	1	0

Then $(X, \cdot)$ is a (strongly) extensional applicative structure. Define $[\![\ ]\!]$ inductively as in Corollary 19. Although $(X, \cdot)$ is total, $[\![\ ]\!]$ will be partial.

Consider the partial functions $\phi(c, b, a) := \mathrm{k} \cdot c \cdot b \cdot a$ and $\psi(c, b, a) := [\![zzy(zzy(zzyx))]\!]_{\rho(z:=c)(y:=b)(x:=a)} = ccb(ccb(ccba))$. Rather tediously, the values of these functions are calculated in this table:

c	b	a	ϕ	ψ
k or 0 or 1	k	k	0	0
	k	0	0	0
	k	1	1	1
	0	k	0	0
	0	0	0	0
	0	1	1	1
	1	k	0	0
	1	0	1	1
	1	1	0	0

Hence by Corollary 19, $[\![h]\!] = [\![\lambda zyx.zzy(zzy(zzyx))]\!]$ is defined and equal to k, and consequently $[\![f]\!] = [\![hh]\!] = \mathrm{kk} = 0$. If $\rho(u) = \rho(x) = 0$ and $\rho(t) = 1$, then

$$\begin{aligned} [\![fu(fu(fu(ftx)))]\!]_\rho &= 1 \\ [\![fu(fu(ft(ftx)))]\!]_\rho &= 0. \end{aligned}$$

By soundness, $fu(fu(fu(ftx))) \neq_{\beta\eta} fu(fu(ft(ftx)))$ $\Rightarrow$ *$Auuut \neq_{\beta\eta} Auutt$.* □

4.5 Towards partial completeness theorems

For partial models, the following completeness theorem holds trivially, since the models can be chosen to be total.

Proposition 20 Completeness: *If $M \neq_\beta M'$, then there is a partial model and ρ for which $[\![M]\!]_\rho$, $[\![M']\!]_\rho$ are defined and $[\![M]\!]_\rho \neq [\![M']\!]_\rho$. If $M \neq_{\beta\eta} M'$, then the model can be chosen to be strongly extensional.* □

Of course much more interesting questions can be asked, *e.g.* how close one can come to a *finite completeness theorem* for partial models? In other words: can every inequality $M \neq_\beta M'$ be demonstrated in a finite partial model? The answer is obviously no, since this would yield a decision procedure for convertibility of lambda terms. It remains open to identify interesting strict subclasses of terms for which a finite completeness property holds.

Further Research

In the introduction, we mentioned the *topological completeness problem*: does there exist a topological model which is complete for β-conversion. This is a long-standing open problem, see *e.g.* [5]. Here, by a topological model we mean a model which arises as a reflexive object in a cartesian closed category of complete partial orders and Scott-continuous functions. Our results imply that if such a model is complete, then the term denotations in it must be discretely ordered. It remains to be seen whether this result can shed some light on the topological completeness problem.

We have also made partial progress with respect to another incompleteness question, namely *order-incompleteness*: Does every lambda theory arise as the theory of a non-trivially partially ordered model? We have characterized order-incomplete lambda theories, or equivalently, absolutely unorderable algebras, syntactically via the presence of generalized Mal'cev operators. The question remains whether, for some $n \geq 1$, such operators are consistent with the lambda calculus, and hence, whether an order-incomplete theory exists. So far, answers are only known for the cases $n = 1$ and $n = 2$.

There are many open questions regarding our *partial syntactical lambda models*. These models seem to resist many of the usual constructions, *e.g.* it is unclear whether they form an interesting category or how to define substructures etc. The soundness properties in Proposition 16 lack an important feature: they do not embody reasoning from additional hypotheses. While this can be easily fixed in the case of soundness for reduction, it is a non-trivial problem in the case of soundness for convertibility, since the latter involves the Church-Rosser property in a crucial way.

Further, it is an interesting question how close one can get, using partial models, to a *finite completeness theorem* for the lambda calculus. With the prospects for such a theorem limited by undecidability considerations, it might nevertheless be possible to identify suitable and interesting classes of terms for which a finite model property holds. In particular, what is the theory of finite partial models, *i.e.* which equations hold in all finite partial models?

Conclusion

This paper provides some basic new facts about ordered models of the type-free lambda calculus. While most known models are ordered, we show that the familiar term algebras are not; they do not allow a partial order. A consequence, with a possible application to the *topological completeness problem*, is that in an ordered model which is complete for β- or $\beta\eta$-conversion, the term denotations must necessarily form a discrete subset.

We also investigate the problem of *order-incompleteness*: Can every model of the lambda calculus be embedded in an orderable one, or equivalently, does every lambda theory arise as the theory of a non-trivially partially ordered model? Towards an answer to this question, we show that a theory, or a model, is order-incomplete if and only if it has a family of generalized Mal'cev operators. Such operators are known from universal algebra, where they are used to characterize permutabilities of congruence relations. In our context, Mal'cev operators are relevant because of their ability to interfere with partial orders. It remains open whether such operators are consistent with the lambda calculus.

We introduce a new notion of finite models for the lambda calculus, called *partial syntactic lambda models*. Partial models can be a convenient tool for reasoning about lambda terms. They can be regarded as a special case of the more natural class of Plotkin's syntactical models of reduction. Unlike in traditional models of the lambda calculus, which are never recursive, term denotations can be easily and effectively calculated in finite models.

Acknowledgments

Thanks to Gordon Plotkin for introducing me to the problem of partial orders on term models, and to Furio Honsell for pointing out the connection with incompleteness theorems. Thanks to Dana Scott, Phil Scott, Peter Freyd and Martin Hyland for discussions, to Andre Scedrov for his supervision, and to the Isaac Newton Institute at Cambridge University for hosting all of the above at various times during the Semantics of Computation Program in Fall 1995.

References

[1] H. P. Barendregt. *The Lambda Calculus, its Syntax and Semantics*. North-Holland, 2nd edition, 1984.

[2] P. Di Gianantonio, F. Honsell, and G. D. Plotkin. Countable non-determinism and uncountable limits. In *CONCUR '94*, SLNCS 836, 1994. See also: Uncountable limits and the Lambda Calculus, *Nordic Journal of Computing* 2, 1995.

[3] J.-Y. Girard. The system F of variable types, fifteen years later. *Theoretical Computer Science*, 45:159–192, 1986.

[4] J. Hagemann and A. Mitschke. On n-permutable congruences. *Algebra Universalis*, 3:8–12, 1973.

[5] F. Honsell and S. Ronchi Della Rocca. An approximation theorem for topological lambda models and the topological incompleteness of lambda calculus. *Journal of Computer and System Sciences*, 45(1), 1992.

[6] M. Hyland. A syntactic characterization of the equality in some models for the lambda calculus. *J. London Math. Soc.*, 12:361–370, 1976.

[7] B. Jacobs, I. Margaria, and M. Zacchi. Filter models with polymorphic types. *Theoretical Computer Science*, 95:143–158, 1992.

[8] A. I. Mal'cev. K obščeĭ teorii algebraičeskih sistem. *Mat. Sb. N. S. 35 (77)*, pages 3–20, 1954.

[9] G. D. Plotkin. A semantics for static type inference. *Information and Computation*, 109:256–299, 1994.

[10] G. D. Plotkin. On a question of H. Friedman. Unpublished, 1995.

[11] D. S. Scott. Relating theories of the λ-calculus. In *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 403–450. Academic Press, London, New York, 1980.

[12] J. D. H. Smith. *Mal'cev Varieties*, volume 554 of *Lecture Notes in Mathematics*. Springer Verlag, 1976.

[13] C. Wadsworth. The relation between computational and denotational properties for Scott's D_∞-models of the lambda-calculus. *SIAM J. Comput.*, 5:488–521, 1976.