All passable games are realizable as monotone set coloring games

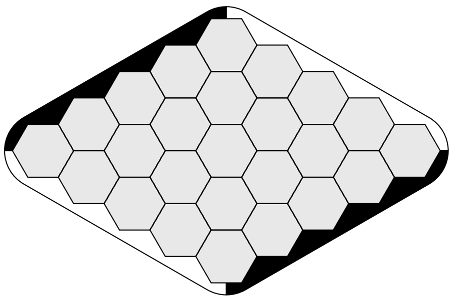
Eric Demer, UCLA
Peter Selinger, Dalhousie University
Kyle Wang, Dalhousie University

Abstract

The class of passable games was recently introduced by Selinger as a class of combinatorial games that are suitable for modelling monotone set coloring games such as Hex. In a monotone set coloring game, the players alternately color the cells of a board with their respective color, and the winner is determined by a monotone function of the final position. It is easy to see that every monotone set coloring game is a passable combinatorial game. Here we prove the converse: every passable game is realizable, up to equivalence, as a monotone set coloring game.

1 Introduction

A class of combinatorial games that is suitable for modelling Hex and other monotone set coloring games was recently introduced in [7]. In a monotone set coloring game [8], the players alternately color the cells of a board with their respective color. When all cells have been colored, the winner is determined by a monotone function of the final board coloring. Hex is perhaps the best-known example of a monotone set coloring game, where the board looks as follows, and the winner is the player (Black or White) who connects the two edges of their own color [5, 6, 3, 4].



Note that the winning condition in Hex is indeed monotone, because the addition of black cells (and the removal of white ones) can only help Black. Also, it is a well-known property of Hex that there are no draws, i.e., one player necessarily wins.

Combinatorial game theory is a formalism for the study of sequential perfect information games. It was introduced by Conway [2] and Berlekamp, Conway, and Guy [1]. In [7], Selinger introduced a class of combinatorial games called "passable" games, and showed that the combinatorial value of every monotone set coloring game is passable. The converse, i.e., whether every (finite) passable game can be realized as the value of a monotone set coloring game, was left as an open question. In this paper, we answer this question positively.

2 Background

2.1 Passable games

We briefly recall some definitions and results that are needed in this paper. For a more comprehensive account, see [7].

Definition 2.1 (Games over a poset)**.** Fix a partially ordered set A, whose elements we call *atoms*. We define a class of combinatorial games as follows:

- For every atom $a \in A$, $[a]$ is a game.
- If L and R are non-empty sets of games, then $\{L \mid R\}$ is a game.

This definition is inductive, i.e., the class of games is the smallest class closed under the above two rules.

We call a game of the form $[a]$ *atomic* and a game of the form $\{L \mid R\}$ *composite*. The elements of L and R are called the game's *left* and *right options*, respectively. We use the standard conventions of combinatorial game theory; for example, we write G^L and G^R for a typical left and right option of G. Note that atomic games do not have options, so if $G = [a]$ is an atomic game, any statement of the form "for all G^L" is vacuously true, and any statement of the form "there exists G^L" is trivially false. We sometimes write a instead of $[a]$ for an atomic game when no confusion arises. The *positions* of a game G are G itself, all of its left and right options, their left and right options, and so on recursively. A game G is called *finite* or *short* if G has finitely many positions. In this paper, we assume that all atom posets have a top element, which we denote $\top$, and a bottom element, which we denote $\bot$.

Definition 2.2 (Order)**.** On the class of games, we define two relations $\leqslant$ and $\lhd$ by mutual recursion as follows.

- $G \leqslant H$ if all three of the following conditions hold:
  1. All left options G^L satisfy $G^L \lhd H$, and
  2. all right options H^R satisfy $G \lhd H^R$, and
  3. if G or H is atomic, then $G \lhd H$.
- $G \lhd H$ if at least one of the following conditions holds:
  1. There exists a right option G^R such that $G^R \leqslant H$, or
  2. there exists a left option H^L such that $G \leqslant H^L$, or
  3. $G = [a]$ and $H = [b]$ are atomic and $a \leqslant b$.

Note that when the games are composite, this definition coincides with the standard definition of $\leqslant$ and $\lhd$ in combinatorial game theory [2]. The only difference is in the treatment of atomic games. See [7] for a detailed explanation of why this definition makes sense.

We say that games G and H are *equivalent*, in symbols $G \approx H$, if $G \leqslant H$ and $H \leqslant G$. We sometimes refer to an equivalence class of games as a *value*, i.e., we say that two games have the same value if and only if they are equivalent. Just like in standard combinatorial game theory, there is a notion of *canonical form* of our games, such that each game is equivalent to a unique canonical form. See [7] for details.

As usual in combinatorial game theory, we say that H is a *left gift horse* for G if $H \lhd G$. The *gift horse lemma* states that for a composite game G, we have $G \approx \{H, G^L \mid G^R\}$ if and only if $H \lhd G$. Of course the dual of this statement is also true, i.e., H is a *right gift horse* for G if $G \lhd H$, and in this case, $G \approx \{G^L \mid G^R, H\}$.

Definition 2.3 (Monotone and passable games)**.** If G is a composite game, we say that a left option G^L is *good* if $G \leqslant G^L$. Dually, we say that a right option G^R is *good* if $G^R \leqslant G$. A game is called *locally monotone* if all of its left and right options are good, and *locally passable* if it is atomic or has at least one good left option or one good right option. A game G is called *monotone* (respectively *passable*) if all positions of G are locally monotone (respectively locally passable).

Intuitively, a good option is one that helps the player who plays it. In a monotone game, all options are good, so making any move is always at least as good as passing. In a passable game, there is at least one good option, so even if passing were allowed, no player would benefit from doing so. This is explained in more detail in [7]. We note that G is locally passable if and only if $G \lhd G$. Trivially, every monotone game is passable. A kind of converse is given by the following theorem:

Theorem 2.4 (Fundamental theorem of passable games [7])**.** *Every passable game is equivalent to a monotone game.*

We also need a few operations on games.

Definition 2.5 (Sum)**.** Let G and H be games over atom posets A and B, respectively. Then their *sum* $G+H$ is a game over the cartesian product poset $A \times B$, and is defined recursively as follows:

- If at least one of G or H is composite:
$$G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}.$$
- If $G = [a]$ and $H = [b]$ are atomic:
$$G + H = [(a, b)].$$

Note that by our convention, the definition simplifies in case one of G, H is atomic and the other is not. For example, if G is atomic and H is not, then G has no left or right options, so $G + H = \{G + H^L \mid G + H^R\}$. As explained in [7], sums do not in general respect equivalence of games; however, when the games are passable, they do. Specifically, if G, G', H, H' are passable and $G \approx G'$ and $H \approx H'$, then $G + H$ and $G' + H'$ are passable and $G + H \approx G' + H'$.

Definition 2.6 (Map operation)**.** Let G be a game over A, and let $f : A \to B$ be a monotone function between atom posets. Then $f(G)$ is a game over B, defined by applying f to all atomic positions in G. More formally, $f(G)$ is defined recursively as follows: $f([a]) = [f(a)]$ and $f(\{G^L \mid G^R\}) = \{f(G^L) \mid f(G^R)\}$.

It is often convenient to combine the sum operation with the map operation. For example, if G, H, K are games over posets A, B, C and $f : A \times B \times C \to D$ is a monotone function, then $f(G + H + K)$ is a game over D.

If S and T are sets of games, we say that S and T are *left equivalent* if for all G^L, G^R, we have $\{S, G^L \mid G^R\} \approx \{T, G^L \mid G^R\}$. Right equivalence is defined dually. We define $\uparrow S = \{\top \mid \{S \mid \bot\}\}$. Then $\uparrow S$ is left equivalent to S. Dually, $\downarrow S = \{\{\bot \mid S\} \mid \top\}$ is right equivalent to S.

2.2 Monotone set coloring games

Fix a partially ordered set A of atoms. Let $\mathbb{B} = \{\top, \bot\}$ be the set of booleans. Here, $\bot$ denotes "false" or "bottom", and $\top$ denotes "true" or "top", with the natural order $\bot < \top$. If X is any set, we write $\mathbb{B}^X$ for the set of functions from X to $\mathbb{B}$. These functions are equipped with the pointwise order, i.e., $f \leqslant g$ if and only if for all $x \in X$, $f(x) \leqslant g(x)$.

Definition 2.7. A *monotone set coloring game* over A is a pair $S = (|S|, \pi_S)$, where $|S|$ is a finite set and $\pi_S : \mathbb{B}^{|S|} \to A$ is a monotone function. The set $|S|$ is called the *carrier* of the game, and its elements are called *cells*. The function π_S is called the *payoff function*. We sometimes write $S : A$ to indicate that S is a monotone set coloring game over A.

If $S : A$ is a monotone set coloring game, a *position* is a map $p : |S| \to \{\top, \bot, \star\}$. Here, $\bot$ indicates a cell occupied by the right player, $\top$ indicates a cell occupied by the left player, and $\star$ indicates an empty cell. When $p(c) = \top$, we also say that the cell c is *colored* with color $\top$, and similarly when $p(c) = \bot$. We write $\star_S$ for the all-empty position. A position is *atomic* if there are no empty cells.

To play the game, the players take turns coloring a cell in their own color, until an atomic position is reached, at which point the payoff function is used to assign a value to the final position. The left player's goal is to maximize the payoff, and the right player's goal is to minimize it.

If p is a non-atomic position, we write p^L for a typical left option, i.e., a position obtained from p by coloring exactly one empty cell with $\top$. Similarly, p^R is a typical right option, i.e., a position obtained from p by coloring exactly one empty cell with $\bot$. Then a monotone set coloring game can be regarded as a combinatorial game in an obvious way, which is made precise in the following definition.

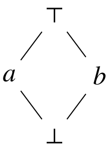
Definition 2.8. Let $S : A$ be a monotone set coloring game. To each position p of S, we associate a combinatorial game $[\![p]\!]$ over A, defined recursively as follows: $[\![p]\!] = [f(p)]$ if p is atomic, and $[\![p]\!] = \{[\![p^L]\!] \mid [\![p^R]\!]\}$ if p is non-atomic. We identify the game S itself with its initial position, i.e., we define $[\![S]\!] = [\![\star_S]\!]$.

Example 2.9. In this paper, we will sometimes describe a monotone set coloring game by a notation such as the following.

$$a : \{○○○●●, ●○●○●, ●●○○○\}, \quad b : \{○○●○○, ○●○●○\},$$

This notation is interpreted as follows. The game has 5 cells, and is over the poset $P_4 = \{\top, a, b, \bot\}$, where $\top$ is the top element, $\bot$ is the bottom element, and a, b are incomparable.

P_4 :



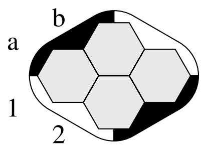
We call the left player "Black" and the right player "White", and we write ● and ○ to denote a cell occupied by Black and White, respectively. If the final position p satisfies $p \geqslant$ ○○○●● or $p \geqslant$ ●○●○● or $p \geqslant$ ●●○○○, then Black achieves at least outcome a. If $p \geqslant$ ○○●○○ or $p \geqslant$ ○●○●○, then Black achieves at least outcome b. If both these conditions hold, then the outcome is $\top$, and if neither holds, the outcome is $\bot$. For example, if the final position is $p =$ ○●○●●, the outcome is $\top$ because $p \geqslant$ ○○○●● and $p \geqslant$ ○○●○○. If the final position is $p =$ ●●○○●, the outcome is a because $p \geqslant$ ●●○○○, but p is not above $p \geqslant$ ○○●○○ or $p \geqslant$ ○●○●○. If the final position is $p =$ ●○○○●, then the outcome is $\bot$.

The combinatorial value of the above game is $\{a, \{\top \mid b\} \mid \{a \mid \bot\}, b\}$, as can be checked by computation.

Example 2.10. The game of Hex is a monotone set coloring game. For example, Hex of size 2×2 can be described as the monotone set coloring game $S : \mathbb{B}$ where $|S| = \{a1, a2, b1, b2\}$ and

$$\pi_S(p) = \begin{cases} \top & \text{if } p(a1) = \top \text{ and } p(a2) = \top, \\ \top & \text{if } p(b1) = \top \text{ and } p(a2) = \top, \\ \top & \text{if } p(b1) = \top \text{ and } p(b2) = \top, \\ \bot & \text{otherwise.} \end{cases}$$

Here again, we identify the color $\top$ with black and $\bot$ with white. The winning condition is exactly the usual one, i.e., Black wins if and only if the black edges are connected by black stones.



Or in the notation of Example 2.9, this game can be concisely described as

$$\top : \{●●○○, ○●●○, ○○●●\}.$$

The combinatorial value of this game is $\{\top \mid \bot\}$, i.e., it is a first-player win.

Remark 2.11. If S is a monotone set coloring game, $[\![S]\!]$ is always a monotone game. This is clear because in a monotone set coloring game, having an extra cell is never bad for a player. In particular, it follows that $[\![S]\!]$ is passable.

3 The main result

In this section, we will prove our main result, namely, that every finite passable game is realizable as a monotone set coloring game.

Definition 3.1. Let G be a combinatorial game over an atom poset A. Then G is *realizable as a monotone set coloring game*, or briefly *realizable*, if there exists a monotone set coloring game $S : A$ such that $G \approx [\![S]\!]$.

Note that by Remark 2.11, every realizable game is equivalent to a passable one. Our main result is the following theorem, which states the converse.

Theorem 3.2 (Realizability theorem for monotone set coloring games)**.** *All finite passable games are realizable as monotone set coloring games.*

The rest of this section is devoted to the proof of this theorem. The proof proceeds in three steps: in Section 3.1, we show that the class of finite passable games can be characterized, up to equivalence, as the smallest class of games containing the atomic games and closed under four operations, which we call gadgets. In Section 3.2, we show that any class of games that contains a small number of specific simple games and is closed under the sum and map operations is automatically closed under the four gadgets. In Section 3.3, we show that the class of games that are realizable as monotone set coloring games satisfies the above conditions, and therefore contains all passable games up to equivalence. This proves the theorem.

We note that Sections 3.1 and 3.2 are not about monotone set coloring games; they contain results about passable games that are of independent interest. Only the results of Section 3.3 are specifically about monotone set coloring games.

3.1 A characterization of finite passable games

Fix an atom poset A. By a *gadget*, we mean an operation that produces a game over A from one or more smaller such games. The following proposition characterizes the class of finite passable games as being the smallest class that contains the atomic games and is closed under four specific gadgets.

Proposition 3.3. *Up to equivalence of games, the class of finite passable games over A is the smallest class $\mathscr{C}$ of games containing the atomic games and closed under the following four gadgets:*

(a) Left forcing gadget: if $G \in \mathscr{C}$, then $\{\top \mid G\} \in \mathscr{C}$.

(b) Right forcing gadget: if $G \in \mathscr{C}$, then $\{G \mid \bot\} \in \mathscr{C}$.

(c) Choice gadget: if $G, H \in \mathscr{C}$, then $\{\{\top \mid G\}, \{\top \mid H\} \mid \{G \mid \bot\}, \{H \mid \bot\}\} \in \mathscr{C}$.

(d) Coupling gadget: if $G, H \in \mathscr{C}$, then $\{G, \{\top \mid H\} \mid \{G \mid \bot\}, H\} \in \mathscr{C}$.

We collectively call the four gadgets of Proposition 3.3 the *primitive gadgets*. By virtue of the proposition, they can be viewed as the basic building blocks of all finite passable games. Before we prove the proposition, we briefly comment on the meaning of the primitive gadgets. The forcing gadgets force one of the players to immediately make a move. The choice gadget allows the player whose turn it is to choose between playing G and H, where it will then be that player's turn again. The coupling gadget is superficially similar to the choice gadget, in that it, too, allows the player whose turn it is to choose between G and H. However, whose turn it will be following this choice does not depend on which player made the choice: it is always the right player's turn if G is chosen and the left player's turn if H is chosen.

We first prove the "easy" direction of Proposition 3.3, namely, that the class of finite passable games has the required closure properties.

Lemma 3.4. *All atomic games are passable, and the class of finite passable games is closed under the four gadgets of Proposition 3.3.*

Proof. Atomic games are passable by definition. Now suppose G and H are passable. Then $\{\top \mid G\}$ is clearly passable, since all its options are passable and $\top$ is a good left option. The case of $\{G \mid \bot\}$ is dual. To show that $K = \{\{\top \mid G\}, \{\top \mid H\} \mid \{G \mid \bot\}, \{H \mid \bot\}\}$ is passable, note that we just proved that all of its options are passable, so it suffices to show that K has a good left option. Since G is passable, we have $G \lhd G$, hence $\{G \mid \bot\} \leqslant G$, hence $K \leqslant \{\top \mid G\}$. Hence $\{\top \mid G\}$ is a good left option of K. To show that $J = \{G, \{\top \mid H\} \mid \{G \mid \bot\}, H\}$ is passable, it likewise suffices to show that there exists a good left option; clearly $\{\top \mid H\}$ is such an option. □

Note that Lemma 3.4 holds "on the nose", i.e., not just up to equivalence.

To finish the proof of Proposition 3.3, we must show that if a class of games has the required closure properties, then it contains all finite passable games up to equivalence. Equivalently, we must show that every finite passable game can be built, up to equivalence, from atomic games by means of the four gadgets. To facilitate the proof, we first show that several other useful gadgets can be constructed from the primitive ones. Throughout the following lemmas, let $\mathscr{C}$ be a class of games that is closed under the four primitive gadgets of Proposition 3.3 and under equivalence.

Lemma 3.5 (One-sided binary choice gadget)**.** *If $G, H \in \mathscr{C}$, then $\{G, H \mid \bot\} \in \mathscr{C}$.*

Proof. Assume $G, H \in \mathscr{C}$. By the right forcing gadget, we have $\{G \mid \bot\} \in \mathscr{C}$ and $\{H \mid \bot\} \in \mathscr{C}$. Applying the choice gadget to $\{G \mid \bot\}$ and $\{H \mid \bot\}$, we get

$$K = \{\{\top \mid \{G \mid \bot\}\}, \{\top \mid \{H \mid \bot\}\} \mid \{\{G \mid \bot\} \mid \bot\}, \{\{H \mid \bot\} \mid \bot\}\} \in \mathscr{C}.$$

Note that $\{\top \mid \{G \mid \bot\}\} = \uparrow G$ is left equivalent to G, and $\{\{G \mid \bot\} \mid \bot\}$ is equivalent to $\bot$, and similarly for H. Therefore, $K \approx \{G, H \mid \bot\}$, and since $\mathscr{C}$ is closed under equivalence, $\{G, H \mid \bot\} \in \mathscr{C}$ as claimed. □

Lemma 3.6 (One-sided n-ary choice gadget)**.** *If $G_1, \ldots, G_n \in \mathscr{C}$, then $\{G_1, \ldots, G_n \mid \bot\} \in \mathscr{C}$.*

Proof. By induction on $n \geqslant 1$. The base cases are the right forcing gadget for $n = 1$ and the one-sided binary choice gadget of Lemma 3.5 for $n = 2$. Now consider $n \geqslant 3$. By the induction hypothesis, $\{G_1, \ldots, G_{n-1} \mid \bot\} \in \mathscr{C}$. Therefore, by the left forcing gadget, $\{\top \mid \{G_1, \ldots, G_{n-1} \mid \bot\}\} = \uparrow(G_1, \ldots, G_{n-1}) \in \mathscr{C}$. By Lemma 3.5, $\{\uparrow(G_1, \ldots, G_{n-1}), G_n \mid \bot\} \in \mathscr{C}$. Since $\uparrow(G_1, \ldots, G_{n-1})$ is left equivalent to $G_1, \ldots, G_{n-1}$, it follows that $\{G_1, \ldots, G_n \mid \bot\} \in \mathscr{C}$ as claimed. □

We need two more lemmas before we can prove Proposition 3.3.

Lemma 3.7 (Coupling lemma)**.** *If $G, H \in \mathscr{C}$ and $K = \{G \mid H\}$ is locally monotone, then $K \in \mathscr{C}$.*

Proof. By definition of local monotonicity, we have $H \leqslant K \leqslant G$. It follows that $\{\top \mid H\} \lhd K$ and $K \lhd \{G \mid \bot\}$. Therefore $\{\top \mid H\}$ and $\{G \mid \bot\}$ are left and right gift horses for K, respectively. Thus K is equivalent to $\{G, \{\top \mid H\} \mid \{G \mid \bot\}, H\}$, which is in $\mathscr{C}$ by the coupling gadget. Since $\mathscr{C}$ is closed under equivalence, we have $K \in \mathscr{C}$. □

The following is a generalization of the coupling lemma to games with more than two options. A game is called *locally semi-monotone* if it has at least one good left option and at least one good right option. Every monotone game is semi-monotone, and every semi-monotone game is passable.

Lemma 3.8 ((n, m)-ary coupling lemma)**.** *If $G_1, \ldots, G_n, H_1, \ldots, H_m \in \mathscr{C}$ and $K = \{G_1, \ldots, G_n \mid H_1, \ldots, H_m\}$ is locally semi-monotone, then $K \in \mathscr{C}$.*

Proof. Let $G = \{G_1, \ldots, G_n \mid \bot\}$ and $H = \{\top \mid H_1, \ldots, H_m\}$. Then $G, H \in \mathscr{C}$ by Lemma 3.6 and its dual. By the forcing gadgets, we also have $\{\top \mid G\} \in \mathscr{C}$ and $\{H \mid \bot\} \in \mathscr{C}$. Let $K' = \{\{\top \mid G\} \mid \{H \mid \bot\}\}$. Note that the left option of K' is $\{\top \mid G\} = \uparrow(G_1, \ldots, G_n)$, which is left equivalent to $G_1, \ldots, G_n$, and similarly for the right option. Hence $K' \approx K$. It is easy to see that K' is locally monotone; namely, since K is semi-monotone, it has some good left option G_i. Then we have $K \leqslant G_i$, and therefore $K \lhd G$, which implies $K \leqslant \{\top \mid G\}$, hence $K' \leqslant \{\top \mid G\}$. The proof of $\{H \mid \bot\} \leqslant K'$ is dual. Therefore, the hypotheses of Lemma 3.7 are satisfied. Hence $K' \in \mathscr{C}$, and therefore $K \in \mathscr{C}$. □

Proof of Proposition 3.3. By Lemma 3.4, the class of finite passable games over A satisfies the closure properties of Proposition 3.3. What we need to show is that it is the smallest such class up to equivalence. So let $\mathscr{C}$ be any class of games containing all atomic games and closed under the four primitive gadgets and under equivalence of games. We must show that all finite passable games belong to $\mathscr{C}$. Since by the fundamental theorem of passable games, every passable game is equivalent to a monotone game, it is sufficient to show that all finite monotone games belong to $\mathscr{C}$. We will show this by induction on games. For the base case, note that all atomic games belong to $\mathscr{C}$ by assumption. For the induction step, consider some monotone game $K = \{G_1, \ldots, G_n \mid H_1, \ldots, H_m\}$. By the induction hypothesis, $G_1, \ldots, G_n, H_1, \ldots, H_m \in \mathscr{C}$. Then $K \in \mathscr{C}$ by Lemma 3.8. □

Remark 3.9. The coupling gadget $\{G, \{\top \mid H\} \mid \{G \mid \bot\}, H\}$ is self-dual. In Proposition 3.3, we could have equivalently replaced it by the following *left-biased coupling gadget*: $\{\{\top \mid G'\}, \{\top \mid H\} \mid G', H\}$. Indeed, in the presence of forcing, the two are easily seen to be equivalent via the substitutions $G' = \{G \mid \bot\}$ and $G = \{\top \mid G'\}$. Thus, while it is an essential feature of the coupling gadget that the player whose turn it is in G or H does not depend on which player makes the choice of whether to play G or H, it is inessential who that player is. Of course there is also a dual *right-biased coupling gadget*.

We end this subsection by pointing out that Proposition 3.3 gives rise to an induction principle for passable games. To show that all passable games have some property, it suffices to show that the property is invariant under equivalence, holds for atomic games, and is closed under the four primitive gadgets. This is usually more convenient than working directly by induction on the definition of passable games, because the latter requires each induction step to use the property of local passability, which is not itself an inductive property.

3.2 Making gadgets from games

In this section, we show that when a class of games is closed under the sum and map operations, then the existence of four specific games is sufficient to guarantee closure under the primitive gadgets of Proposition 3.3.

Let $P_3 = \{\top, a, \bot\}$ be the 3-element linearly ordered poset, and let $P_4 = \{\top, a, b, \bot\}$ be the 4-element poset from Example 2.9. Let A be any poset with top and bottom elements. We define functions $f : P_3 \times A \to A$ and $g : P_4 \times A \times A \to A$ by

$$f(x,y) = \begin{cases} \top & \text{if } x = \top, \\ y & \text{if } x = a, \\ \bot & \text{if } x = \bot \end{cases} \qquad \text{and} \qquad g(x,y,z) = \begin{cases} \top & \text{if } x = \top, \\ y & \text{if } x = a, \\ z & \text{if } x = b, \\ \bot & \text{if } x = \bot. \end{cases}$$

Then f and g are monotone functions. Therefore, if X is a game over P_3 and G is a game over A, then the sum $f(X+G)$ is a well-defined game over A. Similarly, if X is a game over P_4 and G, H are games over A, the sum $g(X + G + H)$ is a well-defined game over A. For brevity, we will henceforth write these sums simply as $X + G$ or $X + G + H$, i.e., the functions f and g will be understood from the context.

Lemma 3.10. *Let G, H be games over A. Then the following hold:*

(a) $\{\top \mid a\} + G \approx \{\top \mid G\}$.

(b) $\{a \mid \bot\} + G \approx \{G \mid \bot\}$.

(c) $\{\{\top \mid a\}, \{\top \mid b\} \mid \{a \mid \bot\}, \{b \mid \bot\}\} + G + H \approx \{\{\top \mid G\}, \{\top \mid H\} \mid \{G \mid \bot\}, \{H \mid \bot\}\}$.

(d) $\{a, \{\top \mid b\} \mid \{a \mid \bot\}, b\} + G + H \approx \{G, \{\top \mid H\} \mid \{G \mid \bot\}, H\}$.

Proof. (a) By induction on G. Let $X = \{\top \mid a\}$ and $K = \{\top \mid G\}$. We want to show $X + G \approx K$. The left options of $X + G$ are:

- $\top + G$. This is easily seen to be equivalent to $\top$.
- $X + G^L$. This is equivalent to $\{\top \mid G^L\}$ by the induction hypothesis.

The right options of $\{\top \mid a\} + G$ are:

- $a + G$. This is easily seen to be equivalent to G.
- $X + G^R$. This is equivalent to $\{\top \mid G^R\}$ by the induction hypothesis.

Therefore, $X + G \approx \{\top, \{\top \mid G^L\} \mid G, \{\top \mid G^R\}\}$. Note that $\{\top \mid G^L\} \lhd \{\top \mid G\}$ and $\{\top \mid G\} \lhd \{\top \mid G^R\}$. Therefore, $\{\top \mid G^L\}$ is a left gift horse and $\{\top \mid G^R\}$ is a right gift horse for K. By the gift horse lemma, it follows that $X + G \approx K$ as claimed.

(b) This is the dual of (a).

(c) By induction on G and H. Let $X = \{\{\top \mid a\}, \{\top \mid b\} \mid \{a \mid \bot\}, \{b \mid \bot\}\}$ and $K = \{\{\top \mid G\}, \{\top \mid H\} \mid \{G \mid \bot\}, \{H \mid \bot\}\}$. We want to show $X + G + H \approx K$. The left options of $X + G + H$ are:

- $\{\top \mid a\} + G + H$. This is equivalent to $\{\top \mid G\}$ by the same argument as in (a).
- $\{\top \mid b\} + G + H$. This is equivalent to $\{\top \mid H\}$ by the same argument as in (a).
- $X + G^L + H$. This is equivalent to $\{\{\top \mid G^L\}, \{\top \mid H\} \mid \{G^L \mid \bot\}, \{H \mid \bot\}\}$ by the induction hypothesis.
- $X + G + H^L$. This is equivalent to $\{\{\top \mid G\}, \{\top \mid H^L\} \mid \{G \mid \bot\}, \{H^L \mid \bot\}\}$ by the induction hypothesis.

The right options are dual. We have $G^L \lhd G$, hence $G^L \leqslant \{\top \mid G\}$, hence $G^L \lhd K$, hence $\{G^L \mid \bot\} \leqslant K$, hence $\{\{\top \mid G^L\}, \{\top \mid H\} \mid \{G^L \mid \bot\}, \{H \mid \bot\}\} \lhd K$. Therefore $X + G^L + H$ is a left gift horse for K. Similarly, $X + G + H^L$ is also a left gift horse for K. The situation for the right options is dual. By the gift horse lemma, it follows that $X + G + H \approx K$ as claimed.

(d) By induction on G and H. Let $X = \{a, \{\top \mid b\} \mid \{a \mid \bot\}, b\}$ and $K = \{G, \{\top \mid H\} \mid \{G \mid \bot\}, H\}$. We want to show $X + G + H \approx K$. The left options of $X + G + H$ are:

- $a + G + H$. This is easily seen to be equivalent to G.
- $\{\top \mid b\} + G + H$. This is equivalent to $\{\top \mid H\}$ by the same argument as in (a).
- $X + G^L + H$. This is equivalent to $\{G^L, \{\top \mid H\} \mid \{G^L \mid \bot\}, H\}$ by the induction hypothesis.
- $X + G + H^L$. This is equivalent to $\{G, \{\top \mid H^L\} \mid \{G \mid \bot\}, H^L\}$ by the induction hypothesis.

The right options are dual. Since $\{G^L, \{\top \mid H\} \mid \{G^L \mid \bot\}, H\} \leqslant \{\top \mid H\}$, we have $\{G^L, \{\top \mid H\} \mid \{G^L \mid \bot\}, H\} \lhd K$. Therefore, $X + G^L + H$ is a left gift horse for K. Similarly, since G is a left option of K, we have $\{G \mid \bot\} \leqslant K$, hence $\{G, \{\top \mid H^L\} \mid \{G \mid \bot\}, H^L\} \lhd K$. Therefore $X + G + H^L$ is also a left gift horse for K. The situation for the right options is dual. By the gift horse lemma, it follows that $X + G + H \approx K$ as claimed. □

Remark 3.11. Although it is tempting to believe that the analogue of Lemma 3.10 holds for all games over P_3 and/or P_4, this is not the case. For example, it is not in general the case that $\{\{\top \mid a\} \mid a\} + G \approx \{\{\top \mid G\} \mid G\}$. In fact, we have $\{\{\top \mid a\} \mid a\} + G \approx G$. We say that a game X over P_3 is a *gadget game* if $X + G$ is equivalent to the game obtained from X by replacing all occurrences of the atom a by G, and similarly for games over P_4. An analogous definition can also be given for P_n with $n > 4$. In particular, Lemma 3.10 states that the games $\{\top \mid a\}$, $\{a \mid \bot\}$, $\{\{\top \mid a\}, \{\top \mid b\} \mid \{a \mid \bot\}, \{b \mid \bot\}\}$, and $\{a, \{\top \mid b\} \mid \{a \mid \bot\}, b\}$ are gadget games. There are many other gadget games too, but for the results of this paper, we only need the above four.

Corollary 3.12. *Let $\mathscr{C}$ be a class of games over posets with top and bottom elements. If $\mathscr{C}$ is closed under equivalence, sums, and the map operation of monotone functions (or at least of the functions f and g above), and if $\mathscr{C}$ contains the four gadget games $\{\top \mid a\}$, $\{a \mid \bot\}$, $\{\{\top \mid a\}, \{\top \mid b\} \mid \{a \mid \bot\}, \{b \mid \bot\}\}$, and $\{a, \{\top \mid b\} \mid \{a \mid \bot\}, b\}$, then $\mathscr{C}$ is closed under the primitive gadgets of Proposition 3.3.*

3.3 Realizability by monotone set coloring games

We want to show that all finite passable games are realizable as monotone set coloring games. Due to Proposition 3.3 and Corollary 3.12, it suffices to show that the class of realizable games contains all atomic games, is closed under the sum and map operations, and contains the four primitive gadget games. We prove each of these properties in turn.

Lemma 3.13. *Let A be a poset and $a \in A$. The atomic game $[a]$ is realizable.*

Proof. The game $[a]$ is trivially realizable as a monotone set coloring game with empty carrier and constant payoff function. □

Lemma 3.14. *Let A and B be posets, and let G and H be games over A and B, respectively. If G and H are realizable, then so is $G + H$.*

Proof. Let $S : A$ and $T : B$ be monotone set coloring games realizing G and H, respectively. We define a new game $S + T$ as follows. Its carrier is the disjoint union of $|S|$ and $|T|$. The payoff function is given by

$$\pi_{S+T}(p) = (\pi_S(p|_S), \pi_T(p|_T)).$$

An easy induction shows that $[\![S + T]\!] = [\![S]\!] + [\![T]\!]$. □

Lemma 3.15. *Let A and B be posets, let G be a game over A, and let $f : A \to B$ be a monotone function. If G is realizable, then so is $f(G)$.*

Proof. Let $S : A$ be a monotone set coloring game realizing G. We define a new game $f(S)$ with the same carrier as S, and payoff function $\pi_{f(S)}(p) = f(\pi_S(p))$. An easy induction shows that $[\![f(S)]\!] = f([\![S]\!])$. □

Lemma 3.16. *The following values are realizable as monotone set coloring games (over P_3 or P_4, as appropriate):*

(a) $\{\top \mid a\}$.

(b) $\{a \mid \bot\}$.

(c) $\{\{\top \mid a\}, \{\top \mid b\} \mid \{a \mid \bot\}, \{b \mid \bot\}\}$.

(d) $\{a, \{\top \mid b\} \mid \{a \mid \bot\}, b\}$.

Proof. One can verify by direct calculation that the following games realize the claimed values. Note that (a) and (b) are duals, and we already encountered the game (d) in Example 2.9.

(a) a : {○}, $\top$: {●}.

(b) a : {●}, $\top$: $\emptyset$.

(c) a : {○●}, b : {●○}.

(d) a : {○○○●●, ●○●○●, ●●○○○}, b : {○○●○○, ○●○●○}. □

Appendix A contains a list of all values over P_4 that are realizable as monotone set coloring games with up to 5 cells.

Proof of Theorem 3.2. By Lemmas 3.13–3.16, the class of realizable games contains all atomic games, is closed under the sum and map operations, and contains the four primitive gadget games. By definition, it is also closed under equivalence. By Corollary 3.12, this class of games is closed under the primitive gadgets, and therefore by Proposition 3.3, it contains all finite passable games. □

4 The size of set coloring gadgets

Theorem 3.2 was only concerned with the existence of monotone set coloring games, and not with their size. We can define the *size* of a monotone set coloring game as the cardinality of its carrier. It is then a natural question to ask about the size of the set coloring games constructed in Theorem 3.2. The following remarks discuss the size of set coloring implementations of the various gadgets, and how they can be improved. We start with the primitive gadgets. For brevity, we say that a game G is *realizable with size p* to mean that it is realizable as a monotone set coloring game of size p.

Proposition 4.1. *If G and H are realizable with size p and q, respectively, then*

(a) $\{\top \mid G\}$ *and* $\{G \mid \bot\}$ *are each realizable with size* $p + 1$,

(b) $\{\{\top \mid G\}, \{\top \mid H\} \mid \{G \mid \bot\}, \{H \mid \bot\}\}$ *is realizable with size* $\max\{p, q\} + 2$, *and*

(c) $\{G, \{\top \mid H\} \mid \{G \mid \bot\}, H\}$ *is realizable with size* $p + q + 5$.

Proof. Parts (a) and (c) require no special proof, as the claimed sizes are just the sizes of the games constructed in the proof of Theorem 3.2. For example, $\{\top \mid G\}$ can be realized as $\{\top \mid a\} + G$ by Lemma 3.10, where $\{\top \mid a\}$ has a carrier of size 1 by the proof of Lemma 3.16, G has a carrier of size p, and the sum has a carrier of size $p + 1$ by the proof of Lemma 3.14. However, there is an important optimization in part (b): it turns out that in the game $\{\{\top \mid a\}, \{\top \mid b\} \mid \{a \mid \bot\}, \{b \mid \bot\}\} + G + H$, it is not actually necessary to take the disjoint union of the carriers of G and H; instead, the ordinary union suffices. Informally, this is because neither player has an incentive to play in G or H before the game $X = \{\{\top \mid a\}, \{\top \mid b\} \mid \{a \mid \bot\}, \{b \mid \bot\}\}$ has reached an atomic position, at which point at most one of G or H needs to be played. More formally, if the carriers of G and H overlap, the game $G + H$ potentially loses some left options of the form $G^L + H$ and $G + H^L$ (which does not matter since they were left gift horses anyway), and instead gains new left options of the form $G^L + H^L$ (namely, when the left player plays in the common carrier of G and H). It is easy to check that in the proof of Lemma 3.10(c), $X + G^L + H^L$ is still a left gift horse for K, and therefore any additional left options caused by overlapping carriers do not change the result. On the other hand, in Proposition 3.10(d), this is not the case, i.e., when $X = \{a, \{\top \mid b\} \mid \{a \mid \bot\}, b\}$, then $X + G^L + H^L$ is not in general a left gift horse for $X + G + H$. □

Next, we consider the size of the one-sided choice gadgets.

Proposition 4.2. *If $G_1, \ldots, G_n$ are realizable with size $p_1, \ldots, p_n$, respectively, then $\{G_1, \ldots, G_n \mid \bot\}$ is realizable with size* $\max\{p_1, \ldots, p_n\} + 2\lceil \log_2 n \rceil + 1$.

Proof. We first claim that if both $G = \{G_1, \ldots, G_k \mid \bot\}$ and $H = \{H_1, \ldots, H_l \mid \bot\}$ are realizable with size p, then $\{G_1, \ldots, G_k, H_1, \ldots, H_l \mid \bot\}$ is realizable with size $p + 2$. Indeed, let $K = \{\{\top \mid G\}, \{\top \mid H\} \mid \{G \mid \bot\}, \{H \mid \bot\}\}$. Then K is realizable with size $p + 2$ by Proposition 4.1(b). Also, by expanding reversible options, it is easily seen that $K \approx \{G_1, \ldots, G_k, H_1, \ldots, H_l \mid \bot\}$.

The proposition then follows by induction. The base case $n = 1$ holds by Proposition 4.1(a). The induction step is an application of the first claim above, halving (or nearly halving, in case n is odd) the number of options in each step. □

Remark 4.3. The realization of the one-sided choice gadget given in Proposition 4.2 admits the following concrete description. Let $k = \lceil \log_2 n \rceil$. The gadget's carrier consists of the union of the carriers of $G_1, \ldots, G_n$, together with $2k + 1$ additional cells $a_1, b_1, \ldots, a_k, b_k, c$. Each pair of cells (a_i, b_i) implements a choice gadget, and we can think of these pairs as the "digits" of a k-bit binary number. The player whose turn it is starts filling in the digits from left to right, and the other player has no choice but to respond in the same digit (or they will lose the game immediately). In this way, the player chooses one of up to 2^k possibilities. After all the digits are chosen, the player whose turn it is plays in cell c. If that player was Left, the game continues in whichever of the components $G_1, \ldots, G_n$ was chosen. If that player was Right, they win immediately.

We now consider the size of the game $\{G \mid H\}$ in the coupling lemma (Lemma 3.7). It turns out that there is a useful optimization in case the game is of the form $\{\{\top \mid G\} \mid \{H \mid \bot\}\}$, so we consider this as a special case.

Proposition 4.4. *Suppose G and H are realizable with size p and q, respectively.*

(a) If $\{G \mid H\}$ is locally monotone, then it is realizable with size $p + q + 5$.

(b) If $\{\{\top \mid G\} \mid \{H \mid \bot\}\}$ is locally monotone, then it is realizable with size $p + q + 5$.

Proof. (a) As shown in the proof of Lemma 3.7, we have $\{G \mid H\} \approx \{G, \{\top \mid H\} \mid \{G \mid \bot\}, H\}$, so the claim follows by Proposition 4.1(c). (b) Let $K = \{\{\top \mid G\} \mid \{H \mid \bot\}\}$. By definition of local monotonicity, we have $\{H \mid \bot\} \leqslant K \leqslant \{\top \mid G\}$. By definition of the order, this implies $H \lhd K$ and $K \lhd G$, so that H and G are left and right gift horses for K, respectively. Therefore K is equivalent to $\{H, \{\top \mid G\} \mid \{H \mid \bot\}, G\}$, and the result follows again by Proposition 4.1(c). □

We can now consider the size of the game in Lemma 3.8.

Proposition 4.5. *Consider a game $K = \{G_1, \ldots, G_n \mid H_1, \ldots, H_m\}$, where $G_1, \ldots, G_n, H_1, \ldots, H_m$ are realizable with size $p_1, \ldots, p_n, q_1, \ldots, q_m$, respectively.*

(a) *If K is locally semi-monotone, then K is realizable with size* $\max\{p_1,\ldots,p_n\} + \max\{q_1,\ldots,q_m\} + 2\lceil\log_2 n\rceil + 2\lceil\log_2 m\rceil + 7$.

(b) *If K is locally passable, then K is realizable with size* $2\max\{p_1,\ldots,p_n,q_1,\ldots,q_m\} + 2\lceil\log_2 n\rceil + 2\lceil\log_2 m\rceil + 10$.

Proof. (a) As in the proof of Lemma 3.8, K can be realized as $\{\{\top \mid G\} \mid \{H \mid \bot\}\}$, where $G = \{G_1,\ldots,G_n \mid \bot\}$ and $H = \{\top \mid H_1,\ldots,H_m\}$. By Proposition 4.2, G is realizable with size $p = \max\{p_1,\ldots,p_n\} + 2\lceil\log_2 n\rceil + 1$, and H is realizable with size $q = \max\{q_1,\ldots,q_m\} + 2\lceil\log_2 m\rceil + 1$. By Proposition 4.4(b), $\{\{\top \mid G\} \mid \{H \mid \bot\}\}$ is realizable with size $p + q + 5$. The result follows.

(b) Since K is passable, it has at least one good left option or right option. Without loss of generality, assume that K has a good left option G_i; the other case is dual. It is easy to see (as in the proof of [7, Lemma 6.7]) that $K' = \{G_1,\ldots,G_n|H_1,\ldots,H_m,\{G_i|\bot\}\}$ is locally semi-monotone and equivalent to K. By part (a), K', and therefore K, is realizable of size $\max\{p_1,\ldots,p_n\} + \max\{q_1,\ldots,q_m,p_i+1\} + 2\lceil\log_2 n\rceil + 2\lceil\log_2(m+1)\rceil + 7$. Since $\lceil\log_2(m+1)\rceil \leqslant \lceil\log_2 m\rceil + 1$, this implies the claim. □

Putting together the above results, we can get an upper bound on the size of the set coloring realizations of monotone and passable games. We define the *depth* of a finite game G in the obvious way, i.e., atomic games have depth 0, and the depth of a composite game is one more than the maximum depth of its options. We define the *branching factor* of G to be the maximum number of left options or right options of any position occurring in G. The following is a quantitative version of Theorem 3.2. As before, A is some fixed atom poset with top and bottom elements.

Proposition 4.6. *Let G be a finite game over A, and let d and b be its depth and branching factor, respectively.*

(a) *If G is monotone, then G is realizable with size* $(2^d - 1)(4\lceil\log_2 b\rceil + 7)$.

(b) *If G is passable, then G is realizable with size* $(2^d - 1)(4\lceil\log_2 b\rceil + 10)$.

Proof. We prove (a) by induction. Let $C = 4\lceil\log_2 b\rceil + 7$. The claim holds for atomic games, since they are realizable with size 0. If $G = \{G_1,\ldots,G_n|H_1,\ldots,H_m\}$ is composite, then by the induction hypothesis, all of its options are realizable with size $p = (2^{d-1} - 1)C$. Applying Proposition 4.5(a), we get that G is realizable with size

$$2p + 2\lceil\log_2 n\rceil + 2\lceil\log_2 m\rceil + 7 \;\leqslant\; 2p + C \;=\; (2^d - 1)C,$$

as claimed. The proof of (b) is almost identical, except using Proposition 4.5(b). □

Remark 4.7. Naively, it would have been possible to obtain a bound on the size of the realization of passable games directly from Proposition 4.6(a), using the fact that by Theorem 2.4, every passable game is equivalent to a monotone one. However, the translation from passable games to monotone games significantly increases the depth of the game, in the worst case by a factor of about 5. This would have resulted in a bound with a much larger base of exponentiation, say, proportional to 32^d instead of 2^d. Conceptually, the reason we were able to obtain the better bound of Proposition 4.6(b) is that while the translation to monotone games increases the depth, it does not add any additional coupling gadgets, i.e., the additional depth entirely comes from options of the form $\{G_1,\ldots,G_n \mid \bot\}$ or $\{\top \mid H_1,\ldots,H_m\}$. In fact, we could define the *coupling depth* of a game, analogous to depth, but such that it increases only when a game has non-trivial left and right options. In this case, the size of the set coloring realization can be bounded as a function of the coupling depth and a small logarithmic factor. The passage from passable to monotone games does not increase the coupling depth.

Remark 4.8. By Proposition 4.2 or Remark 4.3, there are monotone set coloring games such that the number of cells is strictly smaller (even exponentially smaller) than the number of options the left player has in the canonical form. The smallest example we know is the game $G = \{a_1,\ldots,a_8 \mid \bot\}$, where $a_1,\ldots,a_8$ are incomparable atoms. It is realizable with size 7 by Proposition 4.2.

5 Conclusion and future work

In this paper, we proved that all passable games are realizable as monotone set coloring games. Our proof also sheds new light on the structure of passable games: they are generated from the atomic games by four distinct operations: left and right forcing, choice, and coupling. Thus, the passable games form a kind of algebra with four operations. It is an interesting question which equations, if any, these operations satisfy, but we did not pursue this question here. The method we gave in this paper is general, and could in principle be used to show that passable games are realizable by other classes of games, including non-self-dual games such as Shannon games. This is left for future work.

A Appendix: Values over P_4 that are realizable with up to 5 cells

The following is a complete list of all game values over the poset $P_4 = \{\top, a, b, \bot\}$ that are realizable as monotone set coloring game with up to 5 cells. We give a concrete realization of each value. Values that can be obtained by duality (exchanging $\top$ and $\bot$) or isomorphism (exchanging a and b) have been omitted from the list. We use $\emptyset$ to denote the empty set, and ϵ to denote the unique position on 0 cells.

0 cells:

$\top$	$a : \{\epsilon\},\ \ b : \{\epsilon\}$
a	$a : \{\epsilon\},\ \ b : \emptyset$
b	$a : \emptyset,\ \ b : \{\epsilon\}$
$\bot$	$a : \emptyset,\ \ b : \emptyset$

1 cell:

$\{\top \mid a\}$	a : {○}, b : {●}
$\{\top \mid \bot\}$	a : {●}, b : {●}

2 cells: Values of the form $\{\top \mid G\}$ and $\{G \mid \bot\}$, where G is as above, and:

$\{\{\top \mid a\}, \{\top \mid b\} \mid \{a \mid \bot\}, \{b \mid \bot\}\}$	a : {○●}, b : {●○}

3 cells: Values of the form $\{\top \mid G\}$ and $\{G \mid \bot\}$, where G is as above, and:

$\{a, b \mid \bot\}$	a : {○●●, ●○●}, b : {○●●, ●●○}
$\{\top \mid a, \{b \mid \bot\}\}$	a : {○○●}, b : {○●○, ●○●}

4 cells: Values of the form $\{\top \mid G\}$ and $\{G \mid \bot\}$, where G is as above, and:

$\{a \mid a, \{b \mid \bot\}\}$	a : {○○●●, ○●○●, ●○●○}, b : {●●○○}
$\{\{\top \mid a, b\} \mid a, b\}$	a : {○○●●, ●●○○}, b : {○●○●, ●○●○}
$\{a, \{\top \mid a, b\} \mid \bot\}$	a : {○○●●, ○●○●, ●●●○}, b : {○●●○, ●○●●}
$\{\{\top \mid a\} \mid a, \{b \mid \bot\}\}$	a : {○○○●, ○●●○}, b : {●○●○}
$\{\{\top \mid a\}, \{\top \mid b\} \mid \bot\}$	a : {○○○●, ●●●○}, b : {○○●●, ○●●○}
$\{a, \{\top \mid a, \{b \mid \bot\}\} \mid \bot\}$	a : {○○●●, ○●○●}, b : {○●●○, ●○●●}

5 cells: Values of the form $\{\top \mid G\}$ and $\{G \mid \bot\}$, where G is as above, and:

$\{a, b \mid a\}$	a : {○○○○●, ○○○●○}, b : {○○●○●, ○●○●○, ●●●○○}
$\{\{\top \mid a\} \mid a, b\}$	a : {○○○○●, ○○●●○, ●●○○○}, b : {○●○●○}
$\{a, b, \{\top \mid a, b\} \mid \bot\}$	a : {○○○●●, ○○●○●, ○●●●○}, b : {○○●●●, ○●○●○, ●●○○○}
$\{\{a, \{\top \mid b\} \mid a\} \mid a, b\}$	a : {○○○●●, ○○●○●, ○●○●○, ●●●○○}, b : {○●●○○, ●○○●●, ●○●●○}
$\{a, \{\top \mid b\} \mid a, \{b \mid \bot\}\}$	a : {○○○●●, ○●●○○, ●○●○●}, b : {○○○●○, ○●○○●, ●●●○○}
$\{a, \{\top \mid b\} \mid \{a \mid \bot\}, b\}$	a : {○○○●●, ●○●○●, ●●○○○}, b : {○○●○○, ○●○●○}
$\{a, \{b, \{\top \mid a\} \mid b\} \mid \bot\}$	a : {○○○●●, ○○●○●, ●●○●○}, b : {○●●○○, ●●○○○}
$\{a, \{\top \mid a, b\} \mid a, \{b \mid \bot\}\}$	a : {○○○●●, ○○●○●, ○●○●○}, b : {○●●○○, ●○●○●}
$\{a, \{\top \mid b\} \mid \{a \mid a, \{b \mid \bot\}\}\}$	a : {○○○●●, ○○●○●, ○●○○●, ●○○●○, ●○●○○}, b : {○○○●●, ○●○●○, ●●○○○}
$\{\top \mid \{a \mid a, \{b \mid \bot\}\}, \{b \mid \bot\}\}$	a : {○○○●●, ○○●○●, ○●○○●, ●○○●○}, b : {○○●○●, ○○●●○, ●●○○○}
$\{a, \{\top \mid b\} \mid \{a \mid \bot\}, \{b \mid \bot\}\}$	a : {○○○●●, ○○●○●, ○●○●○}, b : {○●●○○, ●○●○○}
$\{a, \{\top \mid b\} \mid a, \{b \mid b, \{a \mid \bot\}\}\}$	a : {○○○●●, ○●●○○, ●○●○●}, b : {○○●●○, ○●○○●, ●○○●○, ●●○○○}
$\{a, b \mid \{a, \{\top \mid a, \{b \mid \bot\}\} \mid \bot\}\}$	a : {○○○○●, ○○●●○, ○●○●○}, b : {○○●●●, ○●●●○, ●○●○●, ●●○○○}
$\{\top \mid a, \{b, \{\top \mid b, \{a \mid \bot\}\} \mid \bot\}\}$	a : {○○○○●, ○○●●○, ●●○●○}, b : {○○●○●, ○●○●○, ●●○○○}
$\{\{\top \mid a, b\}, \{\top \mid a, \{a, b \mid \bot\}\} \mid \bot\}$	a : {○○○●●, ○●●○○, ●○●○●}, b : {○○●●○, ●○○●●, ●○●○●, ●●○○●}
$\{\{a, \{\top \mid b\} \mid a\}, \{b, \{\top \mid a\} \mid b\} \mid \bot\}$	a : {○○○●●, ○○●○●, ●●○●○}, b : {○●○●●, ●○●○○, ●●○○○}
$\{a, \{\top \mid b\} \mid \{a \mid \bot\}, \{b \mid b, \{a \mid \bot\}\}\}$	a : {○○○●●, ○●●○○, ●○●○●}, b : {○●○●○, ●○○○●, ●●○○○}
$\{a, \{b, \{\top \mid a\} \mid b\} \mid \{a \mid \bot\}, \{b \mid \bot\}\}$	a : {○○○●●, ○○●○●, ○●○●○}, b : {●○●○○, ●●○○○}
$\{\{a, \{\top \mid b\} \mid a\}, \{\top \mid a, \{b \mid \bot\}\} \mid \bot\}$	a : {○○○●●, ○○●○●, ○●○●○}, b : {○○●●●, ●○●○○, ●●○●○}
$\{\{\top \mid a, \{a, \{\top \mid b\} \mid \bot\}\} \mid a, \{b \mid \bot\}\}$	a : {○○○●●, ○○●○●, ●●○○○}, b : {○●○●○, ●○●○○}
$\{\top \mid \{a \mid a, \{b \mid \bot\}\}, \{a, \{\top \mid a, b\} \mid \bot\}\}$	a : {○○○●●, ○○●○●, ○○●●○, ○●○○●, ●○○●○, ●●●○○}, b : {○○●○●, ●○●●○, ●●○○○}
$\{\{\top \mid a\}, \{\top \mid b\} \mid \{a \mid a, \{b \mid \bot\}\}, \{b \mid \bot\}\}$	a : {○○○●●, ○○●○●, ○●○●○}, b : {○●●○○, ●○○○●, ●○○●○}
$\{\{a, \{\top \mid b\} \mid a\}, \{\top \mid a, \{a, \{\top \mid b\} \mid \bot\}\} \mid \bot\}$	a : {○○○●●, ○○●○●, ○●○●○, ●●●○○}, b : {○○●●●, ●○●○○, ●●○●○}
$\{a, \{\top \mid b, \{b, \{\top \mid a\} \mid \bot\}\} \mid \{a \mid \bot\}, \{b \mid \bot\}\}$	a : {○○○●●, ○○●○●, ○●○●○}, b : {○●●○○, ●○●○●, ●○●●○}
$\{\{a, \{\top \mid b\} \mid a\}, \{\top \mid a, \{a, \{\top \mid b\} \mid \bot\}\} \mid a, b\}$	a : {○○○●●, ○○●○●, ○●○●○, ●○●○○}, b : {○●○●●, ●○●○●, ●●○○○}
$\{\top \mid a, \{b \mid b, \{a \mid \bot\}\}, \{b, \{\top \mid b, \{a \mid \bot\}\} \mid \bot\}\}$	a : {○○○●●, ○●●○○, ●○●○●}, b : {○○●●●, ○●○●○, ●○○○●, ●○○●○, ●○●○○}
$\{\{\{\top \mid a\} \mid a, \{b \mid \bot\}\}, \{\{\top \mid b\} \mid b, \{a \mid \bot\}\} \mid \bot\}$	a : {○○●●●, ○●○●●, ●○●○●, ●●○●○}, b : {●○●●○, ●●○○●, ●●●○○}
$\{\{\top \mid a\}, \{\top \mid b\} \mid \{a \mid a, \{b \mid \bot\}\}, \{b \mid b, \{a \mid \bot\}\}\}$	a : {○○○●●, ○○●○●, ○●○●○}, b : {○●●○○, ●○○○●, ●●○○○}
$\{\{\top \mid a, \{a, \{\top \mid b\} \mid \bot\}\}, \{\top \mid b, \{b, \{\top \mid a\} \mid \bot\}\} \mid \bot\}$	a : {○○○●●, ○●●○○, ●○●○●}, b : {○○●●○, ○●○●●, ●●○○○}
$\{\{\top \mid a, \{a, \{\top \mid b\} \mid \bot\}\}, \{\top \mid b, \{b, \{\top \mid a\} \mid \bot\}\} \mid \{a \mid \bot\}, \{b \mid \bot\}\}$	a : {○○○●●, ○●●○○, ●○●○●}, b : {○○●●○, ○●○○●, ●●○●○}

References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning Ways for Your Mathematical Plays*. A. K. Peters, 2nd edition, 2001.

[2] J. H. Conway. *On Numbers and Games*. A. K. Peters, 2nd edition, 2001.

[3] M. Gardner. *The Scientific American Book of Mathematical Puzzles and Diversions*. Simon and Schuster, New York, 1959.

[4] R. B. Hayward and B. Toft. *Hex: The Full Story*. CRC Press, 2019.

[5] P. Hein. Vil de laere Polygon? *Politiken*, December 26, 1942. Translated in [4].

[6] J. Nash. Some games and machines for playing them. Technical Report D-1164, Rand Corporation, Feb. 1952.

[7] P. Selinger. On the combinatorial value of Hex positions. Available from arXiv:2101.06694, Jan. 2021.

[8] J. van Rijswijck. *Set Colouring Games*. PhD thesis, Department of Computing Science, University of Alberta, 2006.