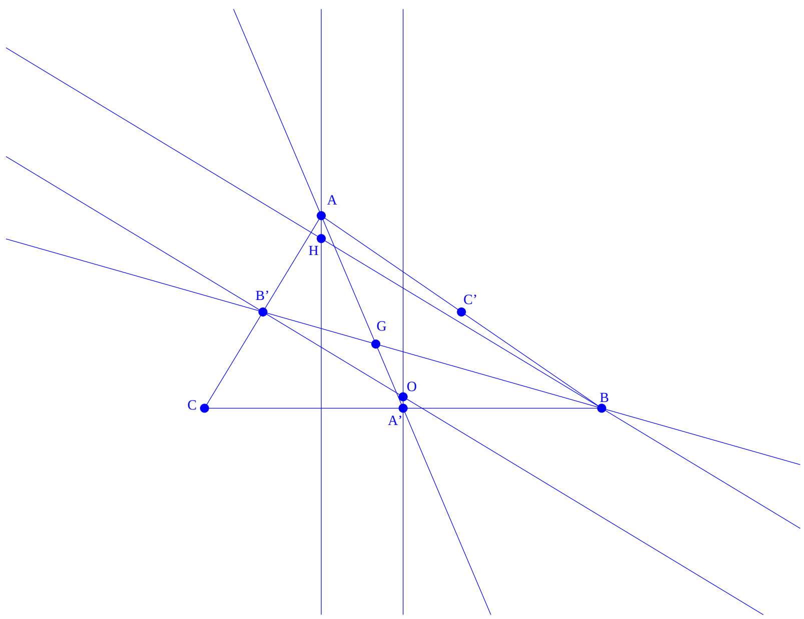
Math 3790
Sept 20th

1 The Euler Line

Consider triangle ABC where we call the bisecting poing of $\bar{BC}$ A', the bisecting point of $\bar{CA}$ B', and the bisecting point of $\bar{AB}$ C'. We need the following definitions:

- The *orthocentre* H is the intersection of the altitudes that go through A and B; that is the line that goes through A and is perpendicular to $\bar{BC}$ and similar for B.
- The *centroid* is the intersection of $B\bar{B}'$ and $A\bar{A}'$ (a line that goes from a vertex of a triangle to the bisecting point of the opposite side is called a *median*
- The *circumcentre* O is the intersection of the perpendicular bisectors of $\bar{AC}$ and $\bar{BC}$.



In the mid 1700's Euler discovered that points H, O, and G are always on the same line (this would eventually be called *the Euler Line*). That is to say the H, O, and G are *collinear*. Additionally he proved $|\bar{HG}| = 2|\bar{GO}|$

In fact Euler proved that triangles AHG and GOA' are similar and then claimed that he was done.

Let's verify Euler's result. We need to prove the following four statements (which we will do in groups).

1. $|\bar{AG}| = 2|\bar{GA'}|$
2. $|\bar{AH}| = 2|\bar{A'O}|$
3. $\angle HAG = \angle OA'G$
4. Once we establish (1), (2), and (3), we're done! In other words, how does it follow that H, G, and O must be collinear, with $|\bar{HG}| = 2|\bar{GO}|$.

Theorem 1.1. *Take three points A, B, and C, and a line that goes through B and separates A and C. Then A, B, C are collinear if and only if the non-adjacent angles of the long segments generated are equal.*

Let us go over a few Euclidean geometry basics.

Definition Two triangles ABC and DEF are considered similar if $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$.

Two triangle ABC and DEF are similar if and only if we have that

$$\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|CA|}{|FD|}.$$

Definition Two distinct lines x and y, in a plane, are parallel if they do not intersect. If we think of lines as subsets of the plane that means $x \cap y = \emptyset$.

Euclidean geometry begins with the idea that given a line x and a point P not on x (sometimes said not incident to x) there is a unique line y that is parallel to x and incident to P.

Theorem 1.2. *If x_1, x_2, x_3 are three distinct lines in a plane such that x_1 is parallel to x_2 and x_2 is parallel to x_3 then x_1 is parallel to x_3.*

Proof. Take a point $P \in x_3$ and let x_3' be the unique line that intersects x_1 and is incident to P. Assume for contradiction that $x_3' \neq x_3$. Then since x_3 is the unique line incident to P and parallel to x_2 we have that x_2 intersects x_3' at a point Q. Then we have that x_3' is the unique line parallel to x_1 and is incident to Q. However x_2 is also incident to Q and parallel to x_1 so $x_2 = x_3'$. But that means x_2 and x_3 are both incident to P which contradicts the fact that x_2 and x_3 is parallel. □

Theorem 1.3. *Given points A, B, $|\bar{AB}| = 0$ if and only if $A = B$.*

Theorem 1.4. *The three medians of a triangle ABC intersect at a common point G. Furthermore, $|\bar{XG}| = 2|\bar{X'G}|$, for all $X \in \{A, B, C\}$.*

Proof. We have actually already done the majority of the proof. Let's call the intersection of $\bar{A'A}$ and $\bar{B'B}$ point G and the intersection of $\bar{A'A}$ and $\bar{C'C}$ point G'. If we can show that $|\bar{G'G}| = 0$ then we've shown that they are actually the same point. Well G is between A and A' and so is G'. So

$$|\bar{G'G}| = \left||\bar{GA}| - |\bar{G'A}|\right|.$$

Well $|\bar{AA'}| = |\bar{AG}| + |\bar{GA'}|$ since G is a midpoint of $\bar{AA'}$. And we showed earlier that $|\bar{AG}| = 2|\bar{GA'}|$ so

$$\begin{aligned} |\bar{AA'}| &= 2|\bar{GA'}| + |\bar{GA'}| \\ &= 3|\bar{GA'}| \end{aligned}$$

or in other words

$$|\bar{GA'}| = |\bar{AA'}|/3.$$

This last fact is based completely on the two facts that G is a midpoint of $\bar{AA'}$ and $|\bar{AG}| = 2|\bar{GA'}|$. But G' is also a midpoint of $|\bar{AG}| = 2|\bar{GA'}|$ and our proof that $|\bar{AG}| = 2|\bar{GA'}|$ gives us that $|\bar{AG'}| = 2|\bar{G'A'}|$ if we replace B with C. So

$$|\bar{G'A'}| = |\bar{AA'}|/3.$$

Giving us that

$$\begin{aligned} |\bar{G'G}| &= \big||\bar{GA}| - |\bar{G'A'}|\big| \\ &= \big|\bar{AA'}|/3 - |\bar{AA'}|/3\big| \\ &= 0 \end{aligned}$$

So $G = G'$.
Finally do $\bar{CC'}$ and $\bar{BB'}$ intersect at G also? Yes, since we know that $G \in \bar{CC'}$ and $G \in \bar{BB'}$, and two distinct lines can have at most one intersection. □

Definition We call this point G the *centroid* or *centre of mass* of triangle ABC.

Another method for proving 3 lines are concurrent is by construction. For example we can prove that the 3 perpendicular bisectors of a triangle are concurrent.

Theorem 1.5. *Take a triangle ABC. Then the three perpendicular bisect at a single point H.*

Proof. We start with $\bar{A'H}$ and $\bar{B'H}$. Since $|\bar{CA'}| = |\bar{BA'}|$ and $90^o = \angle CA'H = \angle BA'H$ we have that triangles $CA'H \cong BA'H$. So $|\bar{BH}| = |\bar{CH}|$. Similarly $|\bar{AH}| = |\bar{CH}|$. So we have that $|\bar{BC'}| = |\bar{AC'}|$ and $|\bar{AH}| = |\bar{BH}|$. Thus $C'AH \cong C'BH$. So $\angle AC'H = \angle BC'H = 90^o$. So $\bar{HC'}$ is the perpendicular bisector of $\bar{AB}$. So H is "incident" to the perpendicular bisector of $\bar{AB}$. □