MATH 2051, Problems in Geometry
Fall 2007
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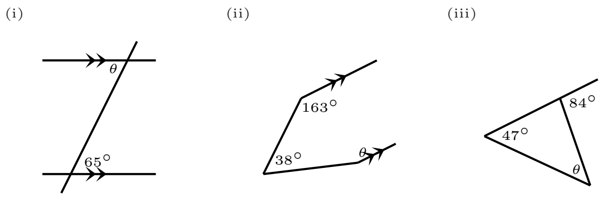
Homework Sheet 1
Due in: Wednesday 19th September, 11:30 PM

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Compulsory questions

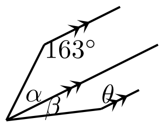
1 *Find the angle* θ *in each of the following diagrams:*



(i) $\theta = 65°$ by alternate angles.

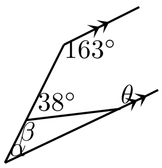
(ii) **Solution 1:**

Draw the line through the vertex with angle $38°$, parallel to the other two lines:



Now by Proposition 1, $\alpha + 163° = 180°$, so $\alpha = 17°$. Also, $\alpha + \beta = 38°$, so $\beta = 21°$. Finally, $\beta + \theta = 180°$ by Proposition 1, so $\theta = 159°$.

Solution 2: Extend one of the non-parallel lines and one of the parallel lines to form a triangle:

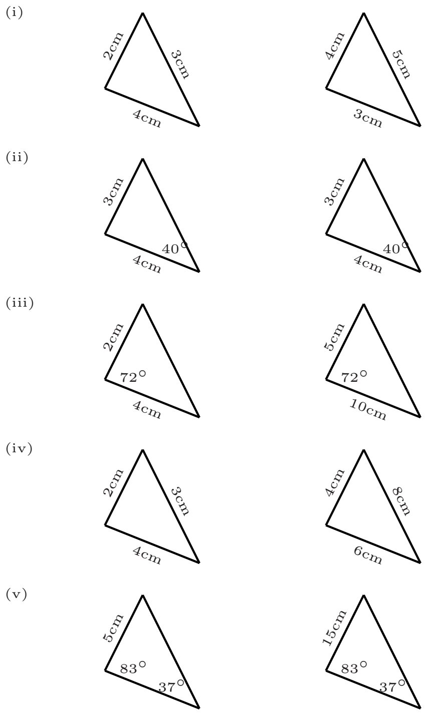


As in (iii), $\alpha + \beta = \theta$. Also, $\beta + 38° = 180°$ by angles on a line, and $\alpha + 163° = 180°$ by Proposition 1, so $\alpha = 17°$, and $\beta = 142°$. Therefore, $\theta = 159°$.

(iii) Let the third angle of the triangle be α. By angles on a line, $\alpha + 84^\circ = 180^\circ$, so $\alpha = 96^\circ$. Therefore, by angles in a triangle, $96^\circ + 47^\circ + \theta = 180^\circ$, so $\theta = 37^\circ$.

Note that the external angle (84°) is equal to the sum of the two opposite internal angles (47° and θ). This is a general fact, and the argument above can easily be turned into a general proof.

2 *Which of the following pairs of triangles are (a) congruent? (b) similar?*



(i) These are not similar, since, for example $\frac{3}{2} \neq \frac{4}{3}$. Therefore, they are not congruent either.

(ii) These are not necessarily congruent (or similar). SAS does not apply, since the angle is not between the two sides. (There are two possible triangles like this – the third sides have lengths about 4.60988cm and 1.51848cm.)

(iii) $\frac{5}{2} = \frac{10}{4}$, so these triangles are similar by SAS.

(iv) These are similar by SSS, since $\frac{4}{2} = \frac{6}{3} = \frac{8}{4}$.

(v) Since angles in a triangle sum to 180°, the unmarked angle in each triangle is 60°. Therefore, the triangles are similar by AAA.

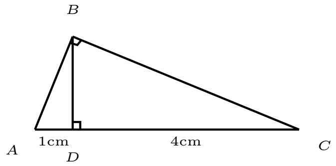
3 *(a) Are any two quadrilaterals with the same side lengths congruent? Give a proof or a counterexample.*

No. For example, a square of side length 1 has all side lengths equal to 1. If we form a quadrilateral by sticking together two equilateral triangles with side length 1, then this forms another quadrilateral with all sides of length 1, which is not congruent to the square.

(b) Are any two quadrilaterals with the same angles similar? Give a proof or a counterexample.

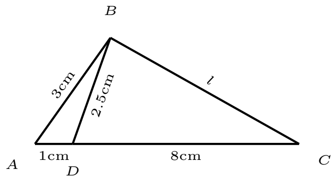
No. For example, a square and a rectangle both have all angles equal to $90°$, but they are not similar.

4 *Let ABC be a right-angled triangle, with the right angle at B. Let D be the point on the side AC such that BD is perpendicular to AC. (See the diagram below.) Suppose AD=1cm, and DC=4cm; find BD.*



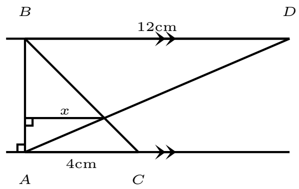
Let $\angle BAD = \alpha$. Then $\angle ABD = 180° - 90° - \alpha = 90° - \alpha$, by angles in a triangle. Therefore, $\angle CDB = \alpha$, and $\angle BCD = 90° - \alpha$. Therefore, by AAA, triangles ABD and BCD are similar. Thus, $\frac{BD}{1\text{cm}} = \frac{4\text{cm}}{BD}$, so $BD^2 = 1 \times 4\text{cm}^2$, therefore, $BD = 2\text{cm}$.

5 *Find the length l in the following diagram:*



Note that $\angle BAD = \angle CAB$, since they are the same angle. Also, $\frac{AC}{AB} = 3 = \frac{AB}{AD}$. Therefore, by SAS, triangles ABD and ACB are similar, with ratio 3. Thus, $\frac{BC}{DB} = 3$, so $BC = 7.5\text{cm}$.

6 *Find the length x in the following diagram:*



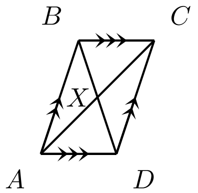
hint:

Call the endpoints of the line segment of length x, E and F, with E on AB and F on BC and AD. Note that EF is parallel to BD and AC.

This should give some useful similar triangles. To use the ratios you get from these triangles, note that $\frac{BE}{BE} + \frac{EA}{BA} = 1$.

7 *A parallellogram is a quadrilateral $ABCD$ in which both pairs of opposite sides are parallel. (i.e. AB and DC are parallel and AD and BC are parallel.)*

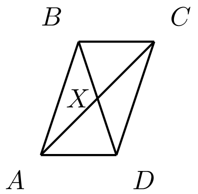
(a) Prove that if $ABCD$ is a parallellogram then $AB = DC$ and $AD = BC$.



Observe that $\angle ABX = \angle CDX$ by alternate angles. Similarly, $\angle BAX = \angle DCX$ by alternate angles. $\angle AXB = \angle CXD$ by vertically opposite angles (or angles in a triangle). Therefore, triangles AXB and CXD are similar by AAA. By the same argument, triangles BXC and DXA are similar.

We therefore have that $\frac{AX}{CX} = \frac{BX}{DX}$ from the first similarity, and $\frac{AX}{CX} = \frac{DX}{BX}$ from the second. This gives $\frac{AX^2}{DX^2} = 1$, so $AX^2 = DX^2$, and since AX and DX are both non-negative, we get that $AX = DX$. Similarly, $BX = CX$. This means that triangle AXB and CXD are congruent, and triangles DXA and BXC are congruent. Therefore, $AB = CD$ and $DA = BC$.

(b) Prove that if $ABCD$ is a quadrilateral such that $AB = DC$ and $AD = BC$, then $ABCD$ is a parallellogram.



We are given that $AB = DC$ and $AD = BC$, and obviously $AD = DA$, so by SSS, triangles ABD and CDB are congruent. This means that $\angle ADB = \angle CBD$, so by the converse of alternating angles, AD and BC are parallel. Similarly, $\angle ABD = \angle CDB$, so by the converse of alternating angles, AB and DC are parallel, so $ABCD$ is a parallelogram.