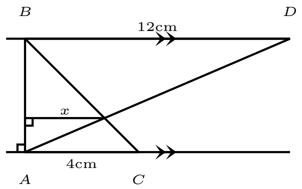
MATH 2051, Problems in Geometry
Fall 2007
Toby Kenney
Homework Sheet 2
Hints and Model Solutions

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Sheet 1

6 *Find the length x in the following diagram:*



hint:

Call the endpoints of the line segment of length x, E and F, with E on AB and F on BC and AD. Note that EF is parallel to BD and AC. This should give some useful similar triangles. To use the ratios you get from these triangles, note that $\frac{BE}{BE} + \frac{EA}{BA} = 1$.

solution:

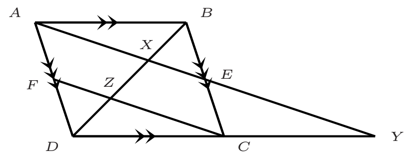
Let X be the point where AD and BC intersect. Let Y be the point where the perpendicular from X to AB meets AB (So YX is the length marked x in the diagram). Now, $\angle ABC = \angle YBX$, since it is the same angle, and by definition, $\angle BYX = \angle BAC = 90°$. Therefore, $\angle BXY = \angle BCA$ by angles in a triangle. Therefore triangles BXY and BCA are similar. Similarly, triangles AXY and ADB are similar.

Hence, $\frac{x}{4\text{cm}} = \frac{BY}{BA}$ and $\frac{x}{12\text{cm}} = \frac{AY}{AB}$. Observe that $\frac{AY}{AB} + \frac{BY}{BA} = \frac{AB}{AB} = 1$, so $\frac{x}{4\text{cm}} + \frac{x}{12\text{cm}} = 1$. This gives $x\left(\frac{1}{4} + \frac{1}{12}\right) = 1$. We calculate $\frac{1}{4} + \frac{1}{12} = \frac{3}{12} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$. Therefore, we get that $x = 3$cm.

Sheet 2

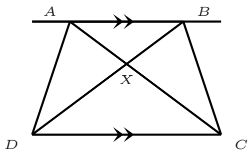
1 *Suppose $ABCD$ is a parallellogram (so AB is parallel to DC and BC is parallel to AD) and E is the midpoint of BC, we draw the line AE, and let X be the point where AE intersects BD. Find the ratio $\frac{BX}{XD}$.*

Let AE and DC meet at Y. Let F be the midpoint of AD, and draw the line FC.



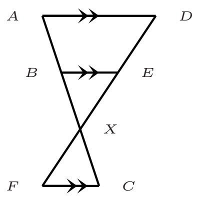
First note that triangles ABE and YCE are congruent by ASA, since $\angle BEA = \angle CEY$ by vertically opposite angles, $\angle ECY = \angle EBA$ by alternate angles, and $BE = CE$. This means that $AE = EY$, so $\frac{AF}{AD} = \frac{AE}{AY} = \frac{1}{2}$, so FE is parallel to AB and DC. Therefore, $CY = AB = DC$ (we know $AB = DC$ from Sheet 1, question 7 (a)), so $\frac{DC}{DY} = \frac{DF}{DA} = \frac{1}{2}$, so FC and AE are parallel, and if Z is the point where DB intersects FC, then $\frac{DZ}{DX} = \frac{1}{2}$. Finally, triangles DZF and BXE are congruent – $\angle DFZ = \angle DAE = \angle BEX$, by alternate angles, and $\angle FDZ = \angle EBX$ by alternate angles, and $FD = \frac{AD}{2} = \frac{BC}{2} = BE$. Therefore, $DZ = ZX = XB$, so $\frac{BX}{XD} = \frac{1}{2}$.

2 *Suppose that lines l_1 and l_2 are parallel. Let A and B be points on l_1 and C and D be points on l_2 such that $AC = BD$, and the lines AC and BD intersect at a point X between l_1 and l_2. Show that $AD = BC$.*



By alternate angles, $\angle BAX = \angle DCX$, and $\angle ABX = \angle CDX$, so triangles ABX and CDX are similar. Therefore, $\frac{BX}{XD} = \frac{AX}{XC}$, so $\frac{AX}{BX} = \frac{CX}{DX}$, therefore, they are both equal to $\frac{AX+CX}{BX+DX} = \frac{AC}{BD} = 1$. This means that $AX = BX$, and $CX = DX$. Now, note that $\angle AXD = \angle BXC$, since they are vertically opposite. Therefore, triangles AXD and BXC are congruent by SAS, so $AD = BC$.

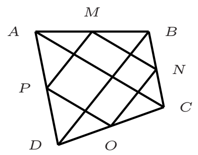
3 *Let l_1, l_2 and l_3 be 3 parallel lines with l_2 between l_1 and l_3. Suppose we also have lines m and n which meet at X between l_2 and l_3. Call the points where m intersects l_1, l_2 and l_3 A, B and C respectively. Call the points where n meets l_1, l_2 and l_3, D, E and F respectively. Show that if $AB = XC$ then $AD = BE + CF$.*



By alternate angles, $\angle XCF = \angle XBE = \angle XAD$ and $\angle XFC = \angle XEB = \angle XDA$, so triangles XCF, XBE and XAD are all similar. Therefore, $\frac{BE}{AD} = \frac{XB}{XA}$, and $\frac{CF}{AD} = \frac{XC}{XA}$, so $\frac{BE+CF}{AD} = \frac{XB+XC}{XA} = \frac{XB+BA}{XA} = 1$, so $BE + CF = AD$.

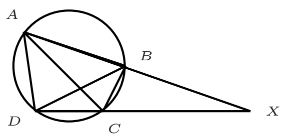
4 *Let $ABCD$ be a quadrilateral. Let M, N, O, and P be the midpoints of AB, BC, CD and DA respectively. Show that $MNOP$ is a parallellogram.*

We want to show that MN and PO are parallel, and that MP and NO are parallel. We will show that MN and OP are both parallel to AC and MP and NO are both parallel to BD.



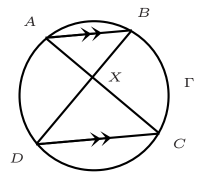
Now $\frac{AM}{AB} = \frac{1}{2} = \frac{AP}{AD}$, so PM and BD are parallel; similarly, $\frac{BM}{BA} = \frac{1}{2} = \frac{BN}{BC}$, so MN and AC are parallel; $\frac{CN}{CB} = \frac{1}{2} = \frac{CO}{CD}$, so NO and BD are parallel; $\frac{DO}{DC} = \frac{1}{2} = \frac{DP}{DA}$, so OP and CA are parallel. Therefore, $MNOP$ is a parallellogram.

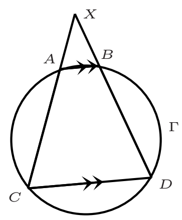
5 *Let A, B, C and D be four points going clockwise round a circle. Suppose AB and DC meet at X, nearer to B than A (and nearer to C than D). If $BX = 2cm$, $BD = 4cm$, and $AC = 5cm$, find CX.*



By angles in the same segment, $\angle CAB = \angle BDC$, and clearly, $\angle AXC = \angle DXB$, since they are the same angle. Therefore, triangles AXC and BXD are similar. Therefore, $\frac{CX}{BX} = \frac{AC}{BD} = \frac{5}{4}$, so $CX = \frac{5\times 2}{4} = \frac{5}{2}$cm.

6 *Let l_1 and l_2 be parallel lines, and let Γ be a circle. Suppose l_1 meets Γ at A and B, and l_2 meets Γ at C and D. Show that $AC = BD$.*

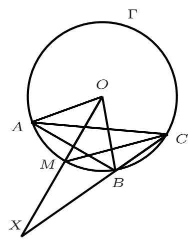




Let AC and BD meet at X, which might be either inside or outside the circle. If X is inside the circle (left-hand diagram) then $\angle XAB = \angle XCD = \angle XBA$, by alternate angles and angles in the same segment. Therefore, triangle XAB is isosceles, and $AX = BX$. Similarly, XDC is isosceles, so $XD = XC$, and so $AC = AX + XC = BX + XD = BD$.

If X is outside the circle (right-hand diagram), then $\angle XAB = \angle XCD = 180° - \angle ABC = \angle XBA$, so triangle XAB is isosceles with $XA = XB$. Similarly, we get that $XC = XD$, so $AC = XC - XA = XD - XB = BD$.

7 *Let A, B and C be three points on a circle Γ, and let M be the point on the arc from A to B not containing C, such that $\angle ACM = \angle BCM$. Let O be the centre of Γ, and let OM meet CB at X. Show that O, C, X and A are concyclic (i.e. they all lie on a common circle).*



We know that $\angle AOM = 2\angle ACM = \angle ACB$, so $\angle AOX = \angle ACX$, so by the converse of angles in the same segment, A, O, C, and X are concyclic. (There is another configuration where X is inside the circle, but the same argument works equally well in that case.)