MATH 2051, Problems in Geometry
Fall 2007
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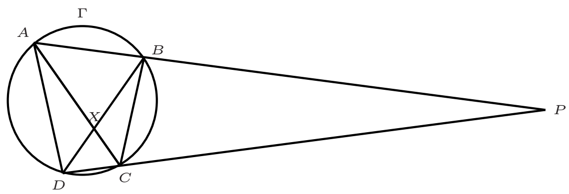
Homework Sheet 3
Hints & Model Solutions

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

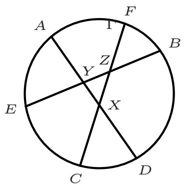
## Compulsory questions

1 *Let $ABCD$ be a cyclic quadrilateral. Let $AB$ and $DC$ meet at $P$ (outside the circle, nearer to $B$ than to $A$, and nearer to $C$ than to $D$). Let $AC$ and $BD$ meet at $X$ (inside the circle) and suppose that $PBXC$ is also cyclic. Prove that $PX$ is a diameter of the circle through $P$, $B$, $X$ and $C$.*




Since $ABCD$ is cyclic, $\angle PBD = 180° - \angle PDB - \angle DPB = 180° - \angle PAC - \angle CAP = \angle PCA$. However, since $PBXC$ is cyclic, $\angle PBD + \angle PCA = 180°$. Therefore, $\angle PBD = \angle PCA = 90°$. Therefore $PX$ is a diameter of the circle through $P$, $C$, $X$, and $B$.

2 *Let $X$, $Y$, and $Z$ be three points inside a circle $\Gamma$, and let the lines $XY$, $YZ$, and $ZX$ meet $\Gamma$ at: $A$ and $D$; $B$ and $E$; and $C$ and $F$ respectively. Suppose that $AY = DX$ and $BZ = EY$. Prove that $CZ = FX$.*



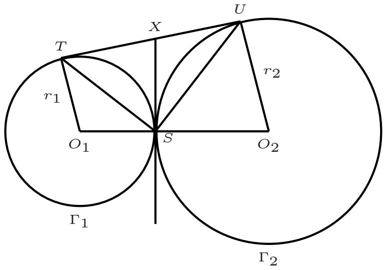

**hint**

Observe that $AX+XD = AD = AY+YD$. Using this we can show $AX = DY$, and similarly $BY = EZ$. Using these facts and the intersecting chord theorem several times, we can get that $CX \times FX = CZ \times FZ$ as well as

$CX + FX = CZ + FZ$. Showing that the only solutions to these equations for $CZ$ and $FZ$ are $CZ = CX, FZ = FX$ and $CZ = FX, FZ = CX$. is then just algebra.

The easiest way to do it is to square the second equation, and subtract off a multiple of the first, to get another square on both sides.

3 *Let circles $\Gamma_1$ and $\Gamma_2$ have centres $O_1$ and $O_2$ and radii $r_1$ and $r_2$ respectively. Suppose that $\Gamma_1$ and $\Gamma_2$ touch externally at a point $S$ (i.e. they meet at $S$, and they have the same tangent there, and they are on opposite sides of the tangent). Suppose that the line $l$ is tangent to $\Gamma_1$ at $T$ and tangent to $\Gamma_2$ at $U$.*




*(a) Show that $\angle TSU = 90°$.*

Let $X$ be the point of intersection of the common tangent at $S$ and the tangent $TU$. Since tangents from a point are equal, we have that $TX = SX = UX$. Therefore, $\angle TSU$ is the angle in a semicircle, so it is $90°$.

*(b) Show that $TU^2 = 4r_1r_2$*

**Solution 1:**

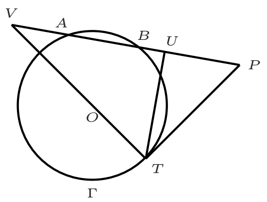
Extend $US$ to meet $TO_1$ at $T'$, and extend $TS$ to meet $UO_2$ at $U'$. Since $\angle TST' = 90°$, $T'$ actually lies on $\Gamma_1$, and similarly, $U'$ lies on $\Gamma_2$. Also, since $\angle O_1TU = 90°$ and $\angle O_2UT = 90°$ by radius and tangent being perpendicular, we have that $\angle SUT = \angle O_1TS$, $\angle TST' = \angle TSU$, and $\angle TT'S = \angle STU$, so triangles $STT'$ and $SUT$ are similar. In the same way, triangles $SUT$ and $SU'U$ are similar. Therefore, $\frac{2r_1}{TU} = \frac{ST}{SU} = \frac{TU}{2r_2}$, and so $4r_1r_2 = TU^2$.

**Solution 2:**

Since $\angle O_1TX = \angle O_1SX = 90°$ by tangent and radius, we have that $\angle TO_1S = 180° - \angle TXS = \angle SXU$, and triangles $TO_1S$ and $UXS$ are both isosceles, so they are similar by SAS. In the same way, $TXS$ and $UO_2S$ are similar, so we have $\frac{r_1}{UX} = \frac{ST}{SU} = \frac{TX}{r_2}$, and since $TX = UX$, we get that $TX^2 = r_1r_2$. Multiplying both sides by 4 gives the required equation.

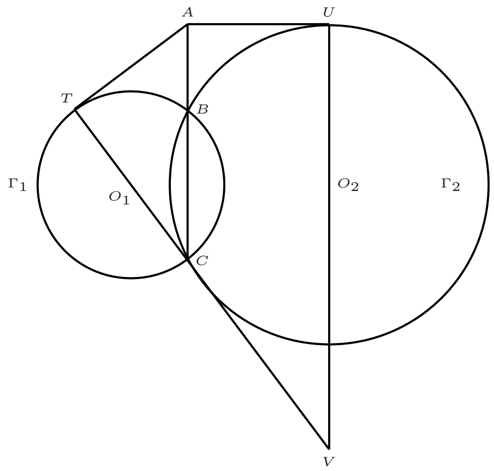
4 *Let $\Gamma$ be a circle, centre $O$, and let $P$ be a point outside $\Gamma$. Let $l$ be a line through $P$, meeting $\Gamma$ at $A$ and $B$, and let $PT$ be a tangent to $\Gamma$ at $T$.*

*Let the line $TO$ meet $AB$ at $V$, and let $U$ be the point on $AB$ such that $\angle TUV = 90°$. Show that $PU \times PV = PA \times PB$.*



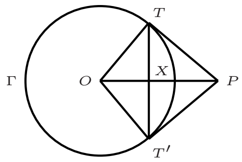

We know that $\angle PTV = 90° = \angle TUP$, and $\angle VPT = \angle TPU$, so since angles in a triangle sum to $180°$, we know that $\angle TVP = \angle UTP$, and triangles $PUT$ and $PTV$ are similar, so $\frac{PT}{PV} = \frac{PU}{PT}$, whence we get that $PU \times PV = PT^2$, and by the intersecting chord and tangent theorem, we know $PT^2 = PA \times PB$.

5 *Let circles $\Gamma_1$ and $\Gamma_2$ have centres $O_1$ and $O_2$ and radii $r_1$ and $r_2$ respectively. Let them intersect at two distinct points $B$ and $C$, and let $A$ be a point on $BC$ outside both circles. Let $AT$ be a tangent to $\Gamma_1$, and let $AU$ be a tangent to $\Gamma_2$, with $T$ and $U$ on $\Gamma_1$ and $\Gamma_2$ respectively. Let the lines $TO_1$ and $UO_2$ meet at $V$. Show that $TV = UV$.*




We know that $AT^2 = AB \times AC = AU^2$, so $AT = AU$ (since they are both non-negative). Also, we know that $\angle ATV = \angle AUV = 90°$, by the angle between a radius and a tangent. Now by Pythagoras' Theorem, $TV^2 = AV^2 - AT^2 = AV^2 - AU^2 = UV^2$, so $TV = UV$.

6 *Let $\Gamma$ be a circle with centre $O$ and radius $r$. Let $P$ be a point outside $\Gamma$. Let $T$ and $T'$ be the points on $\Gamma$, such that the tangents to $\Gamma$ at $T$ and $T'$ pass through $P$. Let $X$ be the point of intersection of $OP$ and $TT'$. Prove that $OP \times OX = r^2$.*




We know that $PT = PT'$, and $OT = OT'$, so triangles $POT$ and $POT'$ are congruent by SSS. Therefore, $\angle TPX = \angle T'PX$, so triangles $TPX$ and $T'PX$ are congruent by SAS. This means that $\angle TXP = \angle T'XP$, so, since their sum is $180°$, they must both be $90°$. Therefore, triangles $POT$ and $TOX$ are similar, since $\angle POT$ and $\angle TOX$ are the same angle, so $\angle OPT = \angle OTX$ by angles in a triangle. This means $\frac{OT}{OX} = \frac{PO}{OT}$, so $PO \times PX = r^2$.