# MATH 2051, Problems in Geometry
## Fall 2007
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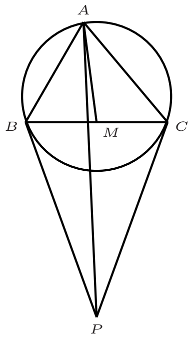
### Homework Sheet 4
### Hints and Model Solutions

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

### Compulsory questions

1 *Let $ABC$ be an acute angled triangle. Let $\Gamma$ be it's circumcircle. Let the tangents to $\Gamma$ at $B$ and $C$ meet at a point $P$. Let $M$ be the midpoint of $BC$. Show that* $\frac{\sin\angle BAP}{\sin\angle PAC} = \frac{\sin\angle CAM}{\sin\angle MAB}$.

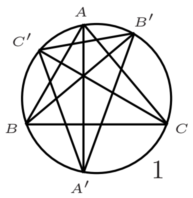



Note that $\angle PBC = \angle BAC$ by the alternate segment theorem, so $\angle PBA = \angle BAC + \angle CBA = 180° - \angle ACB$. Similarly, $\angle PCA = 180° - \angle ABC$. Using the sine rule on triangles $PAB$ and $PAC$, we get $\frac{PA}{\sin\angle PBA} = \frac{PB}{\sin\angle BAP}$ and $\frac{PA}{\sin\angle PCA} = \frac{PC}{\sin\angle CAP}$, and since tangents from a point are equal, $PB = PC$. Therefore, $\frac{\sin\angle CAP}{\sin\angle BAP} = \frac{\sin\angle ACP}{\sin\angle ABP} = \frac{\sin\angle ABC}{\sin\angle ACB}$ (since $\sin(180° - x) = \sin x$).

Applying the sine rule to triangle $ABM$ and $ACM$, we get $\frac{BM}{\sin\angle BAM} = \frac{AM}{\sin\angle ABC}$ and $\frac{MC}{\sin\angle CAM} = \frac{AM}{\sin\angle ACB}$, and by definition, $BM = MC$, so $\frac{\sin\angle BAM}{\sin\angle CAM} = \frac{\sin\angle ABC}{\sin\angle ACB} = \frac{\sin\angle CAP}{\sin\angle BAP}$.

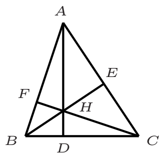
[The line $AP$ is called a *symmedian* of $ABC$. From the result in this question, we can show that $\angle BAP = \angle MAC$. The 3 symmedians of a triangle meet at a point, called the symmedian point of the triangle.]

2 *Let $ABC$ be a triangle. Let the altitudes of the triangle from $A$, $B$, and $C$ meet the circumcircle again at $A'$, $B'$, and $C'$ respectively. Find the angles of triangle $A'B'C'$ in terms of those of $ABC$.*



$\angle C'A'B' = \angle C'A'A + \angle AA'B' = \angle C'CA + \angle ABB' = 90° - \angle BAC + 90° - \angle CAB = 180° - 2\angle BAC$, by angles in the same segment and angles in a triangle (recall that $B'B$ and $C'C$ are perpendicular to $AC$ and $AB$ respectively). Similarly, $\angle A'B'C' = 180° - 2\angle ABC$ and $\angle B'C'A' = 180° - \angle BCA$.

3 *Let $ABC$ be a triangle with orthocentre $H$. Show that if the triangles $ABH$, $ACH$ and $BCH$ have the same perimeter, i.e. if $AB+AH+BH = AC+AH+CH = BC+BH+CH$, then triangle $ABC$ is equilateral (i.e. $AB = AC = BC$).*



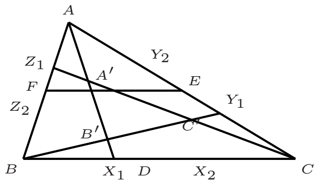

Note that $AH^2 - BH^2 = AF^2 + HF^2 - (FB^2 + HF^2) = AF^2 - BF^2 = AF^2 + FC^2 - (BF^2 + FC^2) = AC^2 - BC^2$. We factor this as $(AH + BH)(AH - BH) = (AC + BC)(AC - BC)$. From the equation $AH + CH + AC = BH + CH + BC$, we get that $AH - BH = BC - AC$, and from $(AH + BH)(AH - BH) = (AC + BC)(AC - BC)$, we get that if $BC - AC \neq 0$, then $AH + BH = -(AC + BC)$, but since they are both lengths, and so positive, this is not possible. Therefore, we must have $AC - BC = 0$, and thus $AC = BC$. Similarly, we get that $AC = AB$, so $ABC$ is equilateral.

4 *Let $ABC$ be a triangle. Let the midpoints of the sides $BC$, $AC$ and $AB$ be $L$, $M$ and $N$ respectively. Let $X_1$ and $X_2$ be the points of trisection of $BC$, with $X_1$ nearer to $B$, i.e. $BX_1 = X_1X_2 = X_2C$. Similarly, let $Y_1$ and $Y_2$ be the points of trisection of $AC$, with $Y_1$ nearer to $C$, and let $Z_1$ and $Z_2$ be the points of trisection of $AB$, with $Z_1$ nearer to $A$.*

*(a) Prove that $CZ_1$ passes through the midpoint of $FE$.*

Let $U$ be the point of intersection of $CZ_1$ and $FE$. We know that $FE$ is parallel to $BC$ (triangles $AFE$ and $ABC$ are similar by SAS, so by corresponding angles, $FE$ and $BC$ are parallel). By corresponding angles, $\angle Z_1FU = \angle Z_1BC$, and $\angle Z_1UF = \angle Z_1CB$, so triangles $Z_1FU$ and $Z_1BC$ are similar. We know that $\frac{Z_1F}{Z_1B} = \frac{1}{4}$, so $FU = \frac{BC}{4} = \frac{FE}{2}$, so $U$ is the midpoint of $FE$.

*Let $A'$ be the point of intersection of $AX_1$ and $CZ_1$, $B'$ the point of intersection of $AX_1$ and $BY_1$, and $C'$ the point of intersection of $BY_1$ and $CZ_1$.*




*(b) Show that $Z_1A' = \frac{Z_1C}{7}$.*

Let $V$ be the midpoint of $DF$. By the same argument as in (a), we know that $V$ lies on $AX_1$. By alternate angles $\angle A'AZ_1 = \angle A'VU$, and $\angle A'Z_1A = \angle A'UV$. Also, $\angle AA'Z_1 = \angle VA'U$ by vertically opposite angles. Therefore, triangles $A'AZ_1$ and $A'VU$ are similar. We know that $UV = \frac{AB}{4}$, and $Z_1A = \frac{AB}{3}$, so we get that $\frac{Z_1A'}{A'U} = \frac{4}{3}$. Thus $\frac{Z_1A'}{Z_1U} = \frac{4}{7}$, and since we already know that triangles $Z_1FU$ and $Z_1BC$ are similar, in the ratio $\frac{1}{4}$, we have that $\frac{Z_1A'}{Z_1C} = \frac{4}{7} \times \frac{1}{4} = \frac{1}{7}$.

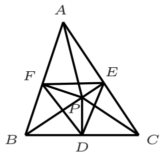
*(c) Show that $A'C' = C'C$.*

Let $W$ be the midpoint of $DE$. By the same argument as in (b), triangles $C'CY_1$ and $C'UW$ are similar in the ratio $\frac{4}{3}$, and therefore, $\frac{C'C}{UC} = \frac{4}{7}$. We also know that $Z_1U = \frac{Z_1C}{4}$, so $\frac{C'C}{Z_1C} = \frac{4}{7}\left(1 - \frac{1}{4}\right) = \frac{3}{7}$. This leaves $\frac{A'C'}{Z_1C} = 1 - \frac{1}{7} - \frac{3}{7} = \frac{3}{7}$, and therefore, $A'C' = C'C$.

5 *Let $ABC$ be an acute angled triangle, and let $P$ be a point in $ABC$. Show that*

$$\frac{AP \times BC}{\sin(\angle BPC - \angle BAC)} = \frac{BP \times AC}{\sin(\angle APC - \angle ABC)} = \frac{CP \times AB}{\sin(\angle APB - \angle ACB)}$$

*[Hint: Let $D$, $E$ and $F$ be the feet of the perpendiculars from $P$ to $BC$, $AC$ and $AB$ respectively. Use the sine rule.]*




**Hint:**

We can show that the quadrilaterals $AEPF$, $CEPD$ and $BDPF$ are cyclic, with diameters $AP$, $CP$ and $BP$ respectively. This allows us to find the lengths $FE$, $DE$ and $DF$, and the angles $\angle DEF$, $\angle EDF$ and $\angle EFD$. We can then use the sine rule on triangle $DEF$.