MATH 2051, Problems in Geometry
Fall 2007
Toby Kenney
Homework Sheet 5
Due in: Wednesday 17th October, 11:30 AM

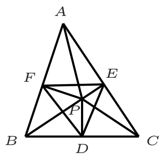
Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Sheet 4

5 *Let ABC be an acute angled triangle, and let P be a point in ABC. Show that*

$$\frac{AP \times BC}{\sin(\angle BPC - \angle BAC)} = \frac{BP \times AC}{\sin(\angle APC - \angle ABC)} = \frac{CP \times AB}{\sin(\angle APB - \angle ACB)}$$

[Hint: Let D, E and F be the feet of the perpendiculars from P to BC, AC and AB respectively. Use the sine rule.]



First note that since $\angle AEP = \angle AFP = 90°$, the quadrilateral $AEPF$ is cyclic, and AP is a diameter of the circumcircle of triangle AEF. Therefore, by the sine rule $\frac{FE}{\sin \angle BAC} = AP$, and if we let R be the radius of the circumcircle of ABC, then by the sine rule, $\frac{BC}{\sin \angle BAC} = 2R$. Therefore, $FE = \frac{AP \times BC}{2R}$. Similarly, $DF = \frac{BP \times AC}{2R}$ and $DE = \frac{CP \times AB}{2R}$.

Now we consider triangle DEF. Since $AFPE$ is cyclic, $\angle PEF = \angle PAF$ and $\angle PED = \angle PCD$. Therefore, $\angle EFD = \angle PAB + \angle PCB$. Now from the quadrilateral $BCPA$, we get that $\angle ABC + \angle BCP + (360° - \angle APC) + \angle PAB = 360°$, and therefore, $\angle FED = \angle PAB + \angle PCB = \angle APC - \angle ABC$. Similarly, $\angle EDF = \angle BPC - \angle BAC$ and $\angle DFE = \angle APB - \angle ACB$.

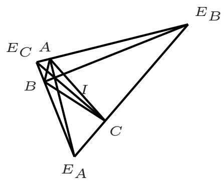
Now we apply the sine rule to triangle DEF, to get

$$\frac{AP \times BC}{2R\sin(\angle BPC - \angle BAC)} = \frac{BP \times AC}{2R\sin(\angle APC - \angle ABC)} = \frac{CP \times AB}{2R\sin(\angle APB - \angle ACB)}$$

and cancelling the common factor of $\frac{1}{2R}$, we get the result.

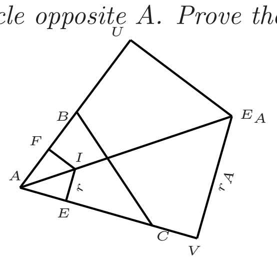
Sheet 5

1 *Let ABC be a triangle with incentre I and excentres E_A, E_B and E_C. Show that I is the orthocentre of triangle $E_AE_BE_C$.*



First we note that E_A and I both lie on the angle bisector of $\angle BAC$, so A, I and E_A are colinear. Therefore, $\angle E_AAE_B = \angle IAE_B$, but since AI and AE_B are the internal and external angle bisectors of $\angle BAC$, $\angle IAE_B = \angle IAC + \angle CAE_B = \frac{1}{2}\angle BAC + \frac{1}{2}\angle CAB' = 90°$. Similarly, $\angle E_BBE_C = 90°$ and $\angle E_CCE_A = 90°$, so E_AA, E_BB and E_CC are the altitudes of $\triangle E_AE_BE_C$, and they meet at I, so I is the orthocentre of $\triangle E_AE_BE_C$.

2 *Let ABC be a triangle. Let s be the semiperimeter; let $a = BC$; and let r_A be the radius of the excircle opposite A. Prove that the area of $\triangle ABC$ is $r_A(s-a)$.*



Since I and E_A both lie on the angle bisector of $\angle BAC$, A, I and E_A are colinear. Let D, E and F be the feet of the perpendiculars from I to AB, AC and BC respectively, and let U, V and W be the feet of the perpendiculars from E_A to AB, AC and BC respectively. Since $\angle AEI = \angle AVE_A = 90°$ and $\angle IAE = \angle E_AAV$, by angles in a triangle, $\angle AIE = \angle AE_AV$, so triangles AIE and AE_AV are similar. Therefore, $\frac{r}{r_A} = \frac{AE}{AV}$. However, since tangents from a point are equal, $EC = CD$, $DB = BF$ and $FA = EA$, $AE+EC+BF = \frac{AB+BC+CA}{2} = s$. Therefore, $AE = s - (CD+BD) = s-a$. Similarly, $CD = s-c$ and $BD = s-b$. Also, since tangents from a point are equal, $CV = CW$, $BU = BW$, and $AV = AU$. Therefore, $AV+AU = b+CV+c+BU = b+c+CW+WB = a+b+c$, so $AU = AV = s$. Therefore, $\frac{r}{r_A} = s - as$, so $rs = r_A(s-a)$, so $r_A(s-a)$ is indeed the area of triangle ABC.

3 *Let ABC be a triangle. Let $a = BC$, $b = AC$ and $c = AB$; and let $A = \angle BAC$, $B = \angle ABC$, and $C = \angle ACB$. Prove that the area of $\triangle ABC$ is $\frac{a^2+b^2+c^2}{4(\cot A + \cot B + \cot C)}$. [Recall that for an angle θ, if we form a triangle ABC with $\angle BAC = 90°$ and $\angle ABC = \theta$, then $\cot\theta = \frac{BA}{CA}$]*

solution 1:

Let the feet of the altitudes be D, E and F, with D on BC, E on AC, and F on AB. We know that $\cot A = \frac{AE}{BE}$, and $\triangle ABC = \frac{b.BE}{2}$, so $\triangle ABC \cot A = \frac{b.AE}{2}$. Doing the same for the other 3 altitudes, we get $2\triangle ABC(\cot A + \cot B + \cot C) = b.AE + c.BF + a.CD$. However, we could instead have used $\cot A = \frac{AF}{CF}$ to get $\triangle ABC \cot A = \frac{c.AF}{2}$, and using the same on the other sides gives $2\triangle ABC(\cot A + \cot B + \cot C) = c.AF + a.BD + b.CE$. Adding the two equations for $2\triangle ABC(\cot A + \cot B + \cot C)$, we get

$$2\triangle ABC(\cot A + \cot B + \cot C) = c.AF + a.BD + b.CE + b.AE + c.BF + a.CD =$$
$$c(AF + FB) + b(AE + EC) + a(BD + DC) = a^2 + b^2 + c^2$$

and dividing through by $4(\cot A + \cot B + \cot C)$ gives us the required formula.

solution 2:

$\triangle ABC \cot A = \frac{1}{2}bc \sin A \cot A = \frac{1}{2}bc \cos A$. Similarly, $\triangle ABC \cot B = \frac{1}{2}ac \cos B$, and $\triangle ABC \cot C = \frac{1}{2}ab \cos C$. Therefore, $\triangle ABC(\cot A + \cot B + \cot C) = \frac{1}{2}(ab \cos C + ac \cos B + bc \cos A)$, but by the cosine rule, $ab \cos C = \frac{1}{2}(a^2 + b^2 - c^2)$, so $\triangle ABC(\cot A + \cot B + \cot C) = \frac{1}{4}(a^2 + b^2 + c^2)$.

4 *Let $ABCD$ be a right angled tetrahedron, i.e. let $\angle BAC = \angle BAD = \angle CAD = 90°$. Prove that $(\triangle BCD)^2 = (\triangle ABC)^2 + (\triangle ABD)^2 + (\triangle ACD)^2$.*

Let $AB = b$, $AC = c$, $AD = d$, $BC = x$, $BD = y$ and $CD = z$. Then by Pythagoras' theorem, we have that $x^2 = b^2 + c^2$, $y^2 = b^2 + d^2$, and $z^2 = c^2 + d^2$. Now we use Hero's formula to get

$$(\triangle BCD)^2 = \frac{(x + y + z)(x + y - z)(x - y + z)(y + z - x)}{16}$$
$$= \frac{((x + y)^2 - z^2)(z^2 - (x - y)^2)}{16}$$
$$= \frac{((\sqrt{b^2 + c^2} + \sqrt{b^2 + d^2})^2 - c^2 - d^2)(c^2 + d^2 - (\sqrt{b^2 + c^2} - \sqrt{b^2 + d^2})^2)}{16}$$
$$= \frac{(2b^2 + 2\sqrt{b^2 + c^2}\sqrt{b^2 + d^2})(2\sqrt{b^2 + c^2}\sqrt{b^2 + d^2} - 2b^2)}{16}$$
$$= \frac{4(b^2 + c^2)(b^2 + d^2) - 4b^4}{16} = \frac{b^2c^2 + b^2d^2 + c^2d^2}{4}$$

On the other hand, since triangles ABC, ABD and ACD are right-angled, $\triangle ABC = \frac{bc}{2}$, $\triangle ABD = \frac{bd}{2}$, and $\triangle ACD = \frac{cd}{2}$, so $(\triangle BCD)^2 = (\triangle ABC)^2 + (\triangle ABD)^2 + (\triangle ACD)^2$.

5 *Let ABC be an acute-angled triangle with othocentre H, and let the feet of the altitudes (the points where the altitude meets the opposite sides) at A, B and C be D, E and F respectively.*

(i) Show that $\frac{HD}{AD} = \cot B \cot C$.

Note that $\angle BHD = 90° - \angle HBD = \angle BCA$ (by angles in a triangle, since $\angle HDB = 90^circ$ and $\angle CEB = 90°$). Therefore, $\frac{HD}{BD} = \cot C$. Also, from triangle ABD, $\frac{BD}{AD} = \cot B$, so $\frac{HD}{AD} = \frac{HD}{BD} \times \frac{BD}{AD} = \cot B \cot C$.

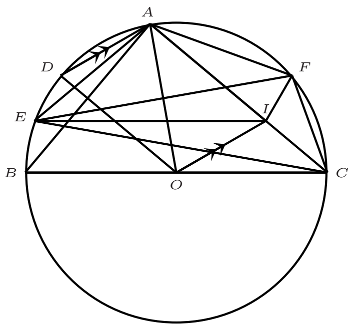
(ii) Deduce that $\cot A \cot B + \cot A \cot C + \cot B \cot C = 1$.

The area of triangle ABC is $\frac{1}{2}BC.AD$, while the area of triangle HBC is $\frac{1}{2}BC.HD$. Therefore, $\frac{\triangle HBC}{\triangle ABC} = \cot B \cot C$. Similarly, $\frac{\triangle AHC}{\triangle ABC} = \cot A \cot C$, and $\frac{\triangle ABH}{\triangle ABC} = \cot A \cot B$. However, $\triangle ABC = \triangle HBC + \triangle AHC + \triangle ABH$, and dividing both sides by $\triangle ABC$, we get $\cot A \cot B + \cot A \cot C + \cot B \cot C = 1$.

[The formula also holds for triangles that are not acute-angled by a similar proof, but in this case, one of the cotangents would be negative, and we would have to subtract the areas.]

Bonus question

6 *Let ABC be a triangle with circumcircle S, and circumcentre O, such that BC is a diameter of S and $\angle AOC > 60°$. Let D be the midpoint of the arc from A to B not containing C. (i.e. D is a point on S between A and B such that $\angle AOD = \angle DOB$.) Let I be the point on AC such that OI is parallel to AD. Let E and F be the points where the perpendicular bisector of OA meets S. Prove that I is the incentre of $\triangle CEF$.*



[IMO 2002 Q.2 – The IMO is an international mathematics competition for schoolchildren. Each country can enter their best 6 contestants. The exam consists of 3 questions in 4 hours 30 minutes, in different areas of mathematics; geometry is a popular area for questions. By IMO standards, this is an easy question.]

By definition, $\angle BOD = \frac{1}{2}\angle BOA = \angle BCA$, since the angle at the centre is twice the angle on the circumference. Therefore, by corresponding angles, the lines OD and CA are parallel, so the quadrilateral $AIOD$ is a parallelogram. Therefore, $AI = DO$ is the radius of the circle. Also, since F is on the perpendicular bisector of AO, $AF = OF$, which is the radius of the circle. Therefore, the triangle AIF is isosceles.

Also, note that triangles AOE and AOF are equilateral, so $\angle AOF = 60°$. With the condition that $\angle AOC > 60°$, this means that F is on the same side of BC as A. This means that $\angle FAC = \frac{1}{2}\angle FOC = \frac{1}{2}(\angle AOC - 60°) = \angle ABC - 30°$. This means that $\angle AFI = 90° - \frac{1}{2}(\angle ABC - 30°)$. Also, $\angle AFE = \frac{1}{2}\angle AOE = 30°$. Therefore, $\angle EFC = 75° - \frac{1}{2}\angle ABC$. On the other hand, $\angle AFC = 180° - \angle ABC$ by angles in a cyclic quadrilateral, so $\angle EFC = 150° - \angle ABC = 2\angle EFI$, so I does lie on the angle bisector of $\angle EFC$.

Also, $\angle ECI = \angle ECA = \frac{1}{2}\angle EOA = 30° = \angle FCI$, so I lies on the angle bisector of $\angle ECF$. Therefore, I is the incentre of triangle ECF.