MATH 2051, Problems in Geometry
Fall 2007

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Homework Sheet 6

Hints & Model Solutions

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Compulsory questions

1 *Given a circle Γ, find a straight-edge and compass construction for its centre, and show that your construction does give the centre of Γ.*

Pick 3 distinct points A, B, and C on Γ. Construct the perpendicular bisectors of AB and BC. The point where they meet is the centre of Γ.

Proof. The point where the perpendicular bisectors of AB and BC meet is the circumcentre of the triangle ABC. But since we chose A, B, and C to be points on Γ, so Γ is the circumcircle of ABC, so the circumcentre of ABC is the centre of Γ. □

2 *Given a circle Γ, and a point P outside Γ, find a straight-edge and compass construction for the points T_1 and T_2 on Γ such that PT_1 and PT_2 are tangent to Γ.*

First construct the centre, O, of Γ as in question 1. Now construct the midpoint M of OP, and draw the circle centre M through O. The points of tangency are the points where this circle intersects Γ.

Proof. Since M is the midpoint of OP, the circle we draw passes through P as well, and OP is a diameter of the circle. Therefore, if T is one of the points of intersection of this circle with Γ, then $\angle OTP = 90°$, since it is the angle in a semicircle. However, OT is a radius of Γ, and since PT is perpendicular to the radius, it is tangent to Γ. □

3 *There is a convex semiregular polyhedron with one pentagonal face and two hexagonal faces meeting at each vertex. How many*

i pentagonal faces

ii hexagonal faces

iii edges

iv vertices

does it have?

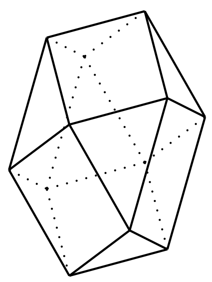
Let F_5 be the number of pentagonal faces, and let F_6 be the number of triangular faces. Let V be the number of vertices, and let E be the number of edges.

Since each pentagonal face has 5 vertices, and each vertex is on exactly one pentagonal face, we have that $V = 5F_5$. Similarly, since each hexagonal face has 6 vertices, and each vertex is shared by 2 hexagonal faces, we have that $2V = 6F_6$. Therefore, we get that $F_6 = \frac{5}{3}F_5$. Finally, we know that there are $\frac{5F_5+6F_3}{2}$ edges, since each edge is shared by 2 faces, and each pentagonal face has 5 edges, while each hexagonal face has 3.

Now we can apply Euler's formula $F_5+F_6-E+V = 2$, and by substituting, we get $F_5 + \frac{5}{3}F_5 - \frac{15}{2}F_5 + 5F_5 = 2$, or $F_5 = 12$. Using the equations from the previous paragraph, this gives $F_6 = 20$, $V = 60$, and $E = 90$.

[This polyhedron can be constructed by cutting all the corners off a regular icosahedron at a point one third of the way along each edge.]

4 *Recall that a cubeoctahedron is a semiregular polyhedron with 8 triangular faces and 6 square faces, with 2 triangular faces and 2 square faces meeting at each vertex. It's vertices are the midpoints of the edges of a cube (or an octahedron).*



(a) What shape are the faces of the dual of the cubeoctahedron? [Hint: they are not regular polygons. Think about the symmetries of the cubeoctahedron (which are the same as symmetries of a cube). You do not have to give all the angles of the shape, just the type of shape]

At each vertex of the cubeoctahedron, we have 2 square faces and two equilateral triangular faces. A face of the dual of the cubeoctahedron is formed by joining the midpoints of these faces. It therefore has 4 vertices, so it is a quadrilateral.

In fact it is a rhombus (a quadrilateral in which all 4 sides have the same length). The easiest way to see this is to note that a cube is symmetric about a plane halfway between a parallel pair of faces, and also symmetric about a plane passing through a diagonally opposite pair of parallel edges. When we view the cubeoctahedron as a cube with thecorners cut off, we see that it is also symmetric about these planes. The first of these planes goes through two vertices of a face of the dual, while the other plane of symmetry goes through the other two vertices. This means that the faces of the dual of the cubeoctahedron are symmetric about the two diagonals. Therefore, all its edges have the same length.

(b) How many faces does it have?

It has a face for every vertex of the cubeoctahedron. The cubeoctahedron has 6 square faces, each of which has 4 vertices, and at each vertex, there are two square faces, so the cubeoctahedron has a total of $\frac{4\times 6}{2} = 12$ vertices. Therefore, its dual has 12 faces.

Alternatively, the vertices of the cubeoctahedron are the midpoints of the edges of the cube, so the cubeoctahedron has 12 vertices because the cube has 12 edges. The dual of the cubeoctahedron therefore has 12 faces.

[This dual is called a rhombic dodecahedron. It can also be formed by putting square pyramids on the faces of a cube.]

5 *Given a convex polyhedron, and a vertex X, let D_X be $360°$ minus the sum of all the angles at X (so for a regular tetrahedron, $D_X = 180°$ for each vertex X, since there are 3 angles of $60°$ at each vertex). Prove that the sum of the D_X over all vertices is $720°$. (For example, for the regular tetrahedron, we have 4 vertices with $D_X = 180°$, so the total is $4 \times 180° = 720°$.) [Hint: the sum of the angles in a polygon with n sides is $180(n-2)°$.]*

The total of all the angles in all the faces is equal to $180°$ times the sum of the number of edges of each face, minus $360°$ times the number of faces. However, the sum of the number of edges of each face is twice the total number of edges, since each edge is on exactly two faces, and so gets counted twice. Therefore, the total of all the angles is $360° \times (E-F)$. The total of all the D_X is therefore $360° \times (V-(E-F)) = 360° \times (V+F-E) = 360° \times 2 = 720°$ by Euler's formula.

6 *In the same way that we constructed duals for semiregular 3-dimensional shapes, we can construct duals for semiregular 4-dimensional shapes. We place a vertex in the centre of each 3-dimensional hyperface, join two vertices by an edge if the 3-dimensional hyperfaces are adjacent, join a collection of vertices with a 2-dimensional face if the 3-dimensional hyperfaces all meet at an edge, and join a collection of vertices with a 3-dimensional hyperface if the corresponding 3-dimensional hyperfaces all meet at a vertex of the original shape.*

What shape are the 3-dimensional hyperfaces of the dual of the hypercube?

4 cubes meet at each vertex of a hypercube. Therefore, each 3-dimensional hyperface of the dual of a hypercube has 4 vertices. It must therefore be a tetrahedron.

Alternatively, the dual to a hypercube has 16 3-dimensional hyperfaces, and 32 2-dimensional faces. Two 3-dimensional hyperfaces meet at each 2-dimensional face. Therefore, each 3-dimensional hyperface has on average 4 faces. However, a hyperface cannot have fewer than 4 faces (since it is a polyhedron) so each hyperface must have exactly 4 faces, so it must be a tetrahedron.