MATH 2051, Problems in Geometry
Fall 2007
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Homework Sheet 7
Due in: Wednesday 14th November, 11:30 AM

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Compulsory questions

1 *(a) Show that the hyperbolic distance from the origin* $(0,0)$ *to the point* $(0,x)$ *is* $2\tanh^{-1} x$. *[Recall that the derivative of* $\tanh x$ *is* $\frac{1}{\cosh^2 x}$*, and that* $1-\tanh^2 x = \frac{1}{\cosh^2 x}$*.]*

The hyperbolic distance from $(0,0)$ to $(0,x)$ is given by $\int_0^x \frac{2dy}{1-y^2}$ (since the x-coordinate is constantly 0. Make the substitution $y = \tanh z$. This gives $\frac{dy}{dz} = \frac{1}{\cosh^2 x} = 1-\tanh^2 x$. Therefore, the integral is $\int_0^{\tanh^{-1} x} \frac{2(1-tanh^2 z)}{1-tanh^2 z} dz = 2\tanh^{-1} x$.

(b) Show that the (hyperbolic) circumference of a circle of hyperbolic radius r *is* $2\pi \sinh r$. *[You may use the formula* $\sinh 2x = 2\sinh x \cosh x$*. Recall also that the circumference of a Euclidean circle of radius* r *is* $2\pi r$*.]*

Apply an isometry to send the centre of the circle to the origin. Now the circle is also a Euclidean circle. Its Euclidean radius is $R = \tanh\left(\frac{r}{2}\right)$ by part (a). Now the hyperbolic circumference is given by integrating $\frac{2\sqrt{dx^2+dy^2}}{1-R^2}$, while the Euclidean circumference is given by integrating $\sqrt{dx^2+dy^2}$, so the hyperbolic circumference is just the Euclidean circumference multiplied by $\frac{2}{1-R^2}$, i.e. it is $\frac{4\pi\tanh\left(\frac{r}{2}\right)}{1-\tanh^2\left(\frac{r}{2}\right)}$ However, from the formula given in part (a), the denominator is $\frac{1}{\cosh^2\left(\frac{r}{2}\right)}$, so the hyperbolic circumference is $4\pi\tanh\left(\frac{r}{2}\right)\cosh^2\left(\frac{r}{2}\right)$. Recall also that $\tanh\left(\frac{r}{2}\right) = \frac{\sinh\left(\frac{r}{2}\right)}{\cosh\left(\frac{r}{2}\right)}$. Substituting this into our formula for the circumference gives that the circumference is $4\pi\sinh\left(\frac{r}{2}\right)\cosh\left(\frac{r}{2}\right) = 2\pi\sinh r$.

2 *Find an explicit formula for an hyperbolic isometry sending the point* $(0,0.5)$ *to* $(0.5,0.5)$*, and the point* $(-0.5,0)$ *to* $(0.3,-0.1)$.

We know that the map $z \mapsto \frac{z-0.5i}{-0.5iz-1}$ sends $0.5i$ to 0 and 0 to $0.5i$. Similarly, $z \mapsto \frac{z-0.5-0.5i}{-0.5iz+0.5z-1}$ sends $0.5+0.5i$ to 0, and 0 to $0.5+0.5i$.

The first of these isometries sends $(-0.5,0)$ to $\frac{-0.5-0.5i}{-1+0.25i} = \frac{(-0.5-0.5i)(-1-0.25i)}{1+0.25^2} = \frac{16}{17}(0.375+0.625i) = \frac{6+10i}{17}$. The second isometry sends $(0.3,-0.1)$ to

$$\frac{0.3-0.1i-0.5-0.5i}{0.1-0.2i-1} = \frac{-2-6i}{-9-2i} = \frac{(2+6i)(9-2i)}{9^2+2^2} = \frac{18+12+54i-4i}{85} = \frac{30+50i}{85} = \frac{6+10i}{17}$$

Therefore, the composite of the two maps sends $0.5i$ to $0.5+0.5i$ and $(-0.5,0)$ to $(0.3,-0.1)$. This composite is

$$\frac{\frac{z-0.5i}{0.5iz-1}-0.5-0.5i}{(0.5-0.5i)\frac{z-0.5i}{0.5iz-1}-1}=\frac{1.25z-0.25iz+0.5}{0.5z-iz+0.75-0.25i}$$

Therefore the map $z\mapsto\frac{1.25z-0.25iz+0.5}{0.5z-iz+0.75-0.25i}$ is an hyperbolic isometry that sends $(0,0.5)$ to $(0.5,0.5)$, and $(-0.5,0)$ to $\left(\frac{31}{106},\frac{-29}{106}\right)$.

3 *Find the midpoint of the hyperbolic line segment from* $(0,0.5)$ *to* $\left(\frac{-30}{58},\frac{41}{58}\right)$. *[You may find the formula* $\tanh 2x=\frac{2\tanh x}{1+\tanh^2 x}$ *and the result in question 1(a) useful.]*

From the formula $\tanh 2x=\frac{2\tanh x}{1+\tanh^2 x}$, we get $\tanh 2x(1+\tanh^2 x)=2\tanh x$, and solving the quadratic gives $\tanh x=\frac{2-\sqrt{4-4\tanh^2(2x)}}{2\tanh 2x}$. This means that the hyperbolic midpoint of the line segment from $(0,0)$ to $(a,0)$ is $\left(\frac{1-\sqrt{1-a^2}}{a},0\right)$.

First we apply the hyperbolic isometry $z\mapsto\frac{z-0.5i}{-0.5iz-1}$, to send the first point to the origin. It sends the point $\left(\frac{-30}{58},\frac{41}{58}\right)$ to $\frac{-30+41i-29i}{15i+20.5-58}=\frac{-30+12i}{-37.5+15i}=0.8$. By the formula we derived above, the midpoint of $(0.8,0)$ and the origin is $\left(\frac{1-\sqrt{0.36}}{0.8},0\right)=(0.5,0)$.

Applying the isometry $z\mapsto\frac{z-0.5i}{-0.5iz-1}$ to $(0.5,0)$ gives $\frac{0.5-0.5i}{-0.25i-1}=\frac{(0.5-0.5i)(0.25i-1)}{1+0.25^2}=\frac{(2-2i)(i-4)}{17}=\frac{-6+10i}{17}$, so the midpoint of the hyperbolic line-segment is $\left(\frac{-6}{17},\frac{10}{17}\right)$.

4 *Show that if A, B and C are three points in the hyperbolic plane such that $AB=AC$ (as hyperbolic distances). Show that A lies on the perpendicular bisector of the hyperbolic line from B to C. [Hint: apply an isometry to make A the origin, then show that the hyperbolic perpendicular bisector is also the Euclidean perpendicular bisector.]*

By applying an hyperbolic isometry, we may assume that A is the origin. This means that the Euclidean distances from A to B and from A to C are equal. The hyperbolic line from B to C is a Euclidean circle. The Euclidean perpendicular bisector of BC is the line through the centre of this circle and the origin A. It is therefore an hyperbolic line, so reflecting in it is an hyperbolic isometry. The point where it meets the hyperbolic line BC is therefore the midpoint of the line segment (since it is fixed by the reflection). Also, since it is a line through the centre of the circle, it meets the circle perpendicularly, so it is also the hyperbolic perpendicular bisector of BC.

5 *The hyperbolic line through the points* $(0,0.5)$ *and* $(0.5,0)$ *is a Euclidean circle. Find the points where this circle meets the boundary of the unit disc.*

Apply the hyperbolic isometry $z \mapsto \frac{z-0.5i}{-0.5iz-1}$, to send the first point to the origin. In question 3, we found that this isometry sends $(0.5, 0)$ to $\left(\frac{-6}{17}, \frac{10}{17}\right)$. The line from the origin to this point meets the boundary of the unit disc at the point $\left(\frac{-6}{\sqrt{136}}, \frac{10}{\sqrt{136}}\right)$ and $\left(\frac{6}{\sqrt{136}}, \frac{-10}{\sqrt{136}}\right)$. Applying the hyperbolic isometry again sends the first of these points to $\frac{6-10i-\sqrt{34}i}{-5-3i-\sqrt{136}}$, which is the point $\left(-\frac{9\sqrt{34}}{170+20\sqrt{34}}, \frac{146+25\sqrt{34}}{170+20\sqrt{34}}\right)$. By the symmetry in the line $y = x$, the other point where the hyperbolic line meets the boundary of the disc will be $\left(\frac{146+25\sqrt{34}}{170+20\sqrt{34}}, -\frac{9\sqrt{34}}{170+20\sqrt{34}}\right)$