MATH 2051, Problems in Geometry
Fall 2007
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Homework Sheet 8
Hints & Model Solutions

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Compulsory questions

1 *(a) We showed that an hyperbolic line is a Euclidean circle perpendicular to the boundary of the unit disc. Show that the origin has power 1 with respect to this circle.*

Recall that the hyperbolic line is given by applying the transformation $z \mapsto \frac{z-a}{\overline{a}z-1}$ to a straight line through the origin. This transformation sends 0 to a, and infinity (which is the limit of the points on the line through the origin) to $\frac{1}{\overline{a}}$. Therefore, the image under $z \mapsto \frac{z-a}{\overline{a}z-1}$ contains the points a and $\frac{1}{\overline{a}}$, and we showed in class that it is a circle. Also, since $\frac{a}{\frac{1}{\overline{a}}} = a\overline{a}$ is real, a and $\frac{1}{\overline{a}}$ lie on the same line through the origin. The distance from the origin to a is $|a|$ (which is $\sqrt{x^2+y^2}$, when $a = x+iy$). Similarly, if $a = x+iy$, then $\frac{1}{\overline{a}} = \frac{a}{a\overline{a}}$, so $\left|\frac{1}{\overline{a}}\right| = \sqrt{\left(\frac{x}{x^2+y^2}\right)^2 + \left(\frac{y}{x^2+y^2}\right)^2} = \sqrt{\frac{x^2+y^2}{(x^2+y^2)^2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{|a|}$, so the power of the origin with respect to this circle is $|a|\left|\frac{1}{\overline{a}}\right| = 1$.

(b) Show that given 3 (Euclidean) circles, Γ_1, Γ_2, and Γ_3, if the radical axis of Γ_1 and Γ_2 passes through the centre of Γ_3, and the radical axis of Γ_1 and Γ_3 passes through the centre of Γ_2, then the radical axis of Γ_2 and Γ_3 passes through the centre of Γ_1.

Let Γ_1 have centre O_1 and radius r_1, Γ_2 have centre O_2 and radius r_2, and Γ_3 have centre O_3 and radius r_3. If O_3 lies on the radical axis of Γ_1 and $Gamma_2$, then it has equal power with respect to Γ_1 and Γ_2, i.e. $O_1O_3^2 - r_1^2 = O_2O_3^2 - r_2^2$. If O_2 also lies on the radical axis of Γ_1 and Γ_3, then we have that $O_2O_3^2 - r_3^2 = O_1O_2^2 - r_1^2$, so

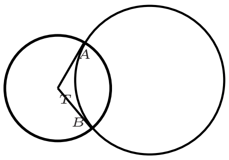
$$O_1O_2^2 - r_2^2 = O_2O_3^2 - r_3^2 + r_1^2 - r_2^2 = O_1O_3^2 - r_1^2 - r_3^2 + r_1^2 = O_1O_2^2 - r_3^2$$

so O_1 lies on the radical axis of Γ_2 and Γ_3.

(c) Let ABC be an hyperbolic triangle such that the (hyperbolic) perpendiculars from A to BC and from B to AC meet at X. Show that the hyperbolic line from C to X is perpendicular to AB. (So X is the orthocentre of the hyperbolic triangle ABC.)

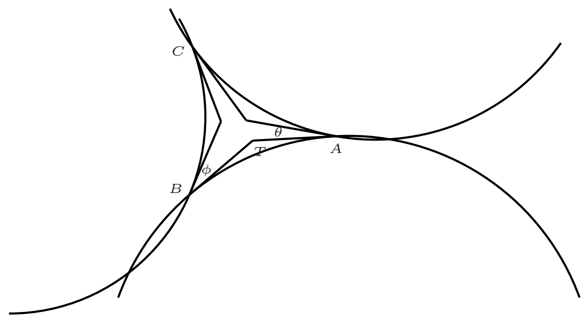
First apply an hyperbolic isometry to send X to the origin. Now the lines AX and BX are the radical axes of the corresponding pairs of circles, since X has power 1 with respect to both circles, and so lies on the radical axis. The hyperbolic line AX now goes through the origin, so it is a Euclidean straight line. It is perpendicular to the hyperbolic line BC, which is a Euclidean circle. A straight line perpendicular to a circle goes through its centre, so the line AX goes through the centre of the hyperbolic line BC, viewed as a Euclidean circle. Similarly, BX goes through the centre of the hyperbolic line AC, viewed as a Euclidean circle. Therefore, by (b), the radical axis of the hyperbolic lines AC and BC, viewed as Euclidean circles, passes through the centre of AB, viewed as a Euclidean circle. This radical axis is the hyperbolic line CX, and is perpendicular to AB, since it goes through the centre of the Euclidean circle.

2 *(a) Hyperbolic circles in the disc model are actually Euclidean circles, and the line joining the hyperbolic centre and the Euclidean centre passes through the origin. Let l be an hyperbolic line not passing through the origin, and A and B be points on l. l is a Euclidean circle; let the (Euclidean) tangents to this circle meet at T. Show that the midpoint of AB is the point where the line from the origin through T meets l. [Hint: consider the hyperbolic circle with diameter AB - it meets l perpendicularly at A and B.]*



The hyperbolic circle Γ with diameter AB is a Euclidean circle, and it meets the hyperbolic line AB perpendicularly at A and B. This means that the radii of this Euclidean circle are tangents to the hyperbolic line at A and B. The Euclidean centre of this circle is therefore T. We know that the line through the hyperbolic centre and the Euclidean centre of a circle goes through the origin, so the line from the origin through T passes through the hyperbolic centre of Γ, which is the midpoint of AB.

(b) Let Γ_1, Γ_2 and Γ_3 be (Euclidean) circles with centres O_1, O_2 and O_3, and radii r_1, r_2, and r_3 respectively. Let Γ_1 and Γ_2 meet at C, Γ_1 and Γ_2 meet at B, and Γ_2 and Γ_3 meet at A. Let T be the point where the tangents to Γ_3 at A and B meet. Suppose that T lies on the radical axis of Γ_1 and Γ_2. Let the angle between Γ_1 and Γ_2 at C (i.e. the angle between their tangents at C) be ψ, the angle between Γ_1 and Γ_3 at B be ϕ, and the angle between Γ_2 and Γ_3 at A be θ. Show that $r_1 \sin\phi = r_2 \sin\theta$.



[Hint: Let the tangent to Γ_3 from A meet Γ_2 again at A', and let the tangent to Γ_3 from B meet Γ_1 again at B'. Show that $AA' = BB'$.]

Since T lies on the radical axis of Γ_1 and Γ_2, we know that T has equal power with respect to Γ_1 and Γ_2. If we let A' be the point where the tangent to Γ_3 at A meets Γ_2 again, and B' be the point where the tangent to Γ_3 at B meets Γ_1 again, then the power of T with respect to Γ_2 is $TA.TA'$, and with respect to Γ_1 is $TB.TB'$, so we have that $TA.TA' = TB.TB'$. Also, since TA and TB are tangents to Γ_3 from the point T, $TA = TB$. Therefore, $TA' = TB'$, and since $AA' = TA' - TA$ and $BB' = TB' - TB$, we get that $AA' = BB'$.

Now $AA' = 2r_2 \sin\alpha$, where α is the angle subtended by the chord AA' at a point on the circumference of Γ_2 (i.e. $\frac{1}{2}\angle AO_2A'$). However, by the alternate segment theorem, $\alpha = \theta$, so $AA' = 2r_2 \sin\theta$. Similarly, $BB' = 2r_1 \sin\phi$, so we have $r_2 \sin\phi = r_2 \sin\theta$.

(c) Assuming that the converse of (b) also holds, deduce that if ABC is an hyperbolic triangle, L, M and N are the midpoints of the sides BC, AC and AB respectively, and AL and BM meet at X, then the (hyperbolic) line CX passes through N.

Apply an hyperbolic isometry that sends X to the origin. Now the hyperbolic lines AB, AC and BC are Euclidean circles - call these circles Γ_1, Γ_2 and Γ_3 respectively. Let the point T be the intersection of the tangents to Γ_3 at B and C. The line AX passes through T, since it is the line from the origin to the midpoint of the hyperbolic line segment AB (see part (a)). Also, AX is the radical axis of Γ_1 and Γ_2. From (b) we therefore get that $r_1 \sin\theta = r_2 \sin\phi$, where r_1 and r_2 are the radii of Γ_1 and Γ_2, and ϕ and θ are the angles between Γ_2 and Γ_3, and between Γ_1 and Γ_3. Similarly, from the fact that BX bisects the hyperbolic line segment AC, we get that $r_3 \sin\phi = r_1 \sin\psi$, where r_3 is the radius of Γ_3, and ψ is the angle between Γ_1 and Γ_2. Therefore,

$$r_2 \sin\psi = \frac{r_1 \sin\theta \sin\psi}{\sin\phi} = \frac{r_3 \sin\phi \sin\psi}{\sin\phi} = r_3 \sin\psi$$

so by the converse of (b), the radical axis of Γ_2 and Γ_3 passes through the point where the tangents to Γ_1 at A and B meet, and since the radical axis passes through the origin, it also passes through N.