MATH 2051, Problems in Geometry
Fall 2007

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Homework Sheet 9

Hints & Model Solutions

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Compulsory questions

1 *Recall that the hyperbolic distance from the origin to a point with modulus r is $2\tanh^{-1} r$. By applying the hyperbolic isometry $z \mapsto \frac{z-a}{\overline{a}z-1}$, we get that the hyperbolic distance between the points z and w is $2\tanh^{-1}\left|\frac{z-w}{\overline{w}z-1}\right|$.*

(a) Let $a = x+iy$. Show that the hyperbolic distance from a to $\overline{a}$ is $2\tanh^{-1}\left(\frac{2y}{\sqrt{(x^2-y^2-1)^2+4x^2y^2}}\right)$.

We just need to show that $\left|\frac{a-\overline{a}}{a^2-1}\right| = \frac{2y}{\sqrt{(x^2-y^2-1)^2+4x^2y^2}}$ Now $\overline{a} = x-iy$, so $a-\overline{a} = 2yi$, and thus $|a-\overline{a}| = 2y$.

$a^2 = (x+iy)^2 = x^2-y^2+2ixy$, so $|a^2-1| = \sqrt{(x^2-y^2-1)^2+4x^2y^2}$.
Now $\left|\frac{a-\overline{a}}{a^2-1}\right| = \frac{|a-\overline{a}|}{|a^2-1|} = \frac{2y}{\sqrt{(x^2-y^2-1)^2+4x^2y^2}}$ as required.

(b) Deduce that the set of points a constant hyperbolic distance d above the x-axis is the set of points satisfying the equation $x^2+y^2+2ky+k^2 = 1+k^2$ (i.e. this set of points is a circle), for $k = \sqrt{\frac{1}{\tanh^2 d}-1} = \sinh d$.

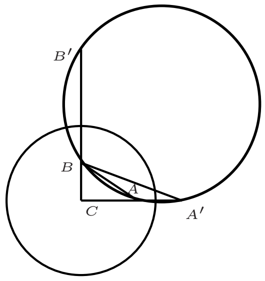
To say that a point is an hyperbolic distance d above the x axis means that it is an hyperbolic distance $2d$ away from its reflection in the x-axis, which is its conjugate. Therefore, if the point is $x+iy$, we have that $2\tanh^{-1}\left(\frac{2y}{\sqrt{(x^2-y^2-1)^2+4x^2y^2}}\right) = 2d$, which gives that $\left(\frac{2y}{\sqrt{(x^2-y^2-1)^2+4x^2y^2}}\right) = \tanh d$. Rearranging this, we get $2y = \tanh d\sqrt{x^4+y^4+1+2x^2y^2-2x^2+2y^2}$, and squaring both sides leads to:

$$\begin{aligned}
4y^2 &= (\tanh^2 d)(x^4+y^4+1+2x^2y^2-2x^2-2y^2+4y^2)\\
4(1-\tanh^2 d)y^2 &= (\tanh^2 d)(1-x^2-y^2)^2\\
2\sqrt{\frac{1-\tanh 2d}{\tanh^2 d}}y &= 1-x^2-y^2\\
x^2+y^2+2ky &= 1\\
x^2+y^2+2ky+k^2 &= 1+k^2
\end{aligned}$$

2 *Let ABC be a right-angled hyperbolic triangle with the right-angle at C. Let a be the (hyperbolic) length of the side BC, and let b be the (hyperbolic) length of the side AC.*

(a) Let $\alpha = \angle CAB$ and $\beta = \angle CBA$. Let $x = \tanh\left(\frac{a}{2}\right)$ and $y = \tanh\left(\frac{b}{2}\right)$. Show that $\tan\alpha = \frac{x(1-y^2)}{y(1+x^2)}$. [Hint: The hyperbolic line through the points x and yi is a circle that also passes through $\frac{1}{x}$ and $\frac{i}{y}$. Use the alternate segment theorem.]

First apply an hyperbolic isometry to send C to the origin, A to the point x, and B to the point yi (By the hyperbolic distance formula, x and yi are the correct distances from the origin). Now the hyperbolic line AB is a Euclidean circle passing through x, yi, $\frac{1}{x}$ and $\frac{i}{y}$ (since the origin has power 1 with respect to the circle).



The angle α is the angle between the chord AA' and the tangent to the circle at A', so by the alternate segment theorem it is equal to the angle subtended by AA' at a point on the circumference, i.e. it is equal to $\angle ABA'$.

We can find $\angle ABA'$ as the argument of the complex number $\frac{x-yi}{\frac{1}{x}-yi}$, which is the same as the argument of $\frac{x-yi}{1-xyi}$, since multiplying by a real number doesn't change the argument. Now $\frac{x-yi}{1-xyi} = \frac{(x-yi)(1+xyi)}{1+x^2y^2} = \frac{x(1+y^2)+y(x^2-1)i}{1+x^2y^2}$, so its argument satisfies $\tan\alpha = \frac{y(1-x^2)}{x(1+y^2)}$.

(b) Show that the area of triangle ABC is $\tan^{-1}\left(\frac{2xy}{1-x^2y^2}\right)$. [You may find the formulae $\tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$ and $\tan\left(\frac{\pi}{2}-\theta\right) = \frac{1}{\tan\theta}$ useful.]

We know that $\tan\alpha = \frac{y(1-x^2)}{x(1+y^2)}$, and by symmetry, $\tan\beta = \frac{x(1-y^2)}{y(1+x^2)}$. Using the tangent of a sum formula, we get

$$\tan(\alpha+\beta) = \frac{\frac{y(1-x^2)}{x(1+y^2)} + \frac{x(1-y^2)}{y(1+x^2)}}{1-\left(\frac{y(1-x^2)}{x(1+y^2)}\right)\left(\frac{x(1-y^2)}{y(1+x^2)}\right)} = \frac{y^2(1-x^4)+x^2(1-y^4)}{2xy(x^2+y^2)} = \frac{1-x^2y^2}{2xy}$$

We know that the area of $\triangle ABC$ is $\frac{\pi}{2}-\alpha-\beta$, so $\tan(\triangle ABC) = \frac{1}{\tan(\alpha+\beta)} = \frac{2xy}{1-x^2y^2}$.

3 *Let ABC be an asymptotic hyperbolic triangle with A at infinity, and angles $\angle ABC = \theta$, $\angle ACB = \phi$.*

(a) Show that the length of the finite side BC of the asymptotic triangle ABC is $l = 2\tanh^{-1}\left(\frac{\sin(\theta+\phi)}{\sin\theta+\sin\phi}\right)$.

[Hint: Assuming B is at the origin, the hyperbolic line from A to C is a Euclidean circle. Find the tangent to this circle at C; let D be the point where this tangent meets the line AB, and use the sine rule on triangle DBC.]

First apply an hyperbolic isometry to send B to the origin and A to 1. The hyperbolic line AC is a Euclidean circle tangent to the real axis at 1. Let the tangent to this circle at C meet the real axis at D.

Since tangents from a point are equal, $DA = DC$. Let $x = DA$, and let $BC = y$. By the hyperbolic distance formula, we know that $l = 2\tanh^{-1} y$. Applying the sine rule to $\triangle DBC$, we get $\frac{x}{\sin\theta} = \frac{1-x}{\sin\phi} = \frac{y}{\sin(\theta+\phi)}$. From the first equation, we get that these 3 fractions are equal to $\frac{1}{\sin\theta+\sin\phi}$, so $y = \frac{\sin(\theta+\phi)}{\sin\theta+\sin\phi}$, and $l = 2\tanh^{-1}\left(\frac{\sin(\theta+\phi)}{\sin\theta+\sin\phi}\right)$.

(b) What is the area of the triangle with largest area with one side of length l?

[You may use the formula:

$$\frac{\sin(\theta+\phi)}{\sin\theta+\sin\phi} = \frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}$$

To explain why the triangle with largest area must be asymptotic, show that any non-asymptotic triangle is contained in an asymptotic triangle.]

If we have a non-asymptotic triangle ABC with BC of (hyperbolic) length l, then we can extend the hyperbolic line BA past A to meet the unit disc at D. Now the hyperbolic triangle DBC contains the triangle ABC, and has a side of length l. Therefore, there is an hyperbolic triangle of maximal area, which is asymptotic.

Let the angles of an asymptotic triangle with side length l be θ, ϕ and 0. By (a), $l = 2\tanh^{-1}\left(\frac{\sin(\theta+\phi)}{\sin\theta+\sin\phi}\right)$. The area of the triangle is $\pi-\theta-\phi$, so we want to minimise $\theta+\phi$ subject to $\frac{\sin(\theta+\phi)}{\sin\theta+\sin\phi} = \tanh\left(\frac{l}{2}\right)$. Using the formula given, we get $\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)} = \tanh\left(\frac{l}{2}\right)$, and we want to minimise $\theta+\phi$ subject to this condition. This is equivalent to maximising $\cos\left(\frac{\theta+\phi}{2}\right)$, since cos is a decreasing function. Since $\tanh\left(\frac{l}{2}\right)$ is constant, this is equivalent to maximising $\cos\left(\frac{\theta-\phi}{2}\right)$. This is achieved by setting $\theta = \phi$. Now we get $\cos\left(\frac{\theta+\phi}{2}\right) = \tanh\left(\frac{l}{2}\right)$, so the area is $\pi - 2\cos^{-1}\left(\tanh\left(\frac{l}{2}\right)\right)$.