MATH 2051, Problems in Geometry
Fall 2007
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Homework Sheet 5
Due in: Wednesday 17th October, 11:30 AM

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Compulsory questions

1 Let ABC be a triangle with incentre I and excentres E_A, E_B and E_C. Show that I is the orthocentre of triangle $E_AE_BE_C$.

2 Let ABC be a triangle. Let s be the semiperimeter; let $a = BC$; and let r_A be the radius of the excircle opposite A. Prove that the area of $\triangle ABC$ is $r_A(s-a)$.

[Hint: Let U and V be the feet of the perpendiculars from E_A to AB and AC. Show that $AU = AV = s$.]

3 Let ABC be a triangle. Let $a = BC$, $b = AC$ and $c = AB$; and let $A = \angle BAC$, $B = \angle ABC$, and $C = \angle ACB$. Prove that the area of $\triangle ABC$ is $\frac{a^2+b^2+c^2}{4(\cot A+\cot B+\cot C)}$. [Recall that for an angle θ, if we form a triangle ABC with $\angle BAC = 90°$ and $\angle ABC = \theta$, then $\cot\theta = \frac{BA}{CA}$

Hint: Multiply both sides by $\cot A+\cot B+\cot C$. Draw in the 3 altitudes AD, BE and CF, and use the fact that $AEDB$, etc. are cyclic.]

4 Let $ABCD$ be a right angled tetrahedron, i.e. let $\angle BAC = \angle BAD = \angle CAD = 90°$. Prove that $(\triangle BCD)^2 = (\triangle ABC)^2+(\triangle ABD)^2+(\triangle ACD)^2$.

5 Let ABC be an acute-angled triangle with othocentre H, and let the feet of the altitudes (the points where the altitude meets the opposite sides) at A, B and C be D, E and F respectively.

(i) Show that $\frac{HD}{AD} = \cot B\cot C$.

(ii) Deduce that $\cot A\cot B + \cot A\cot C + \cot B\cot C = 1$.

Bonus question

6 Let ABC be a triangle with circumcircle S, and circumcentre O, such that BC is a diameter of S, and $\angle AOC > 60°$. Let D be the midpoint of the arc from A to B not containing C. (i.e. D is a point on S between A and B such that $\angle AOD = \angle DOB$.) Let I be the point on AC such that OI is parallel to AD. Let E and F be the points where the perpendicular bisector of OA meets S. Prove that I is the incentre of $\triangle CEF$.