MATH 2051, Problems in Geometry
Fall 2007
Toby Kenney
Homework Sheet 9
Due in: Wednesday 28th November, 11:30 AM

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Compulsory questions

1 Recall that the hyperbolic distance from the origin to a point with modulus r is $2\tanh^{-1} r$. By applying the hyperbolic isometry $z \mapsto \frac{z-a}{\overline{a}z-1}$, we get that the hyperbolic distance between the points z and w is $2\tanh^{-1}\left|\frac{z-w}{\overline{w}z-1}\right|$.

(a) Let $a = x + iy$. Show that the hyperbolic distance from a to $\overline{a}$ is $2\tanh^{-1}\left(\frac{2y}{\sqrt{(x^2-y^2-1)^2+4x^2y^2}}\right)$.

(b) Deduce that the set of points a constant hyperbolic distance d above the x-axis is the set of points satisfying the equation $x^2 + y^2 + 2ky + k^2 = 1 + k^2$ (i.e. this set of points is a circle), for $k = \sqrt{\frac{1}{\tanh^2 d} - 1} = \sinh d$.

2 Let ABC be a right-angled hyperbolic triangle with the right-angle at C. Let a be the (hyperbolic) length of the side BC, and let b be the (hyperbolic) length of the side AC.

(a) Let $\alpha = \angle CAB$ and $\beta = \angle CBA$. Let $x = \tanh\left(\frac{a}{2}\right)$ and $y = \tanh\left(\frac{b}{2}\right)$. Show that $\tan\alpha = \frac{x(1-y^2)}{y(1+x^2)}$. [Hint: The hyperbolic line through the points x and yi is a circle that also passes through $\frac{1}{x}$ and $\frac{i}{y}$. Use the alternate segment theorem.]

(b) Show that the area of triangle ABC is $\tan^{-1}\left(\frac{2xy}{1-x^2y^2}\right)$. [You may find the formulae $\tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$ and $\tan\left(\frac{\pi}{2}-\theta\right) = \frac{1}{\tan\theta}$ useful.]

3 Let ABC be an asymptotic hyperbolic triangle with A at infinity, and angles $\angle ABC = \theta$, $\angle ACB = \phi$.

(a) Show that the length of the finite side BC of the asymptotic triangle ABC is $l = 2\tanh^{-1}\left(\frac{\sin(\theta+\phi)}{\sin\theta+\sin\phi}\right)$.

[Hint: Assuming B is at the origin, the hyperbolic line from A to C is a Euclidean circle. Find the tangent to this circle at C; let D be the point where this tangent meets the line AB, and use the sine rule on triangle DBC.]

(b) What is the area of the triangle with largest area with one side of length l?

[You may use the formula:

$$\frac{\sin(\theta+\phi)}{\sin\theta+\sin\phi} = \frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}$$

To explain why the triangle with largest area must be asymptotic, show that any non-asymptotic triangle is contained in an asymptotic triangle.]