MATH 2051, Problems in Geometry
Fall 2007
Toby Kenney
Mock Final Examination
Time allowed: 3 hours
Calculators not permitted.

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers. This mock exam contains more questions than the final exam will, particularly on hyperbolic geometry, in order to give a better idea of the range of questions that might be asked.

Answer all questions

1 *(a) Let A and B be two points. Let $0 < \lambda < 1$ be a real number. Let P be a point such that $\frac{AP}{BP} = \lambda$. Let $\theta = \angle ABP$. Use the cosine rule on triangle ABP to find a quadratic equation satisfied by BP.*

(b) Since a quadratic equation has at most two solutions, there is at most one other point P' on the line BP such that $\frac{AP'}{BP'} = \lambda$. Show that $\frac{BP+BP'}{2} = \left(\frac{AB}{(1-\lambda^2)}\right)\cos\theta$ and $\frac{BP-BP'}{2} = \left(\frac{AB}{(1-\lambda^2)}\right)\sqrt{\cos^2\theta - (1-\lambda^2)}$.

(c) Let O be the point on AB extended past A, such that $OB = \frac{AB}{1-\lambda^2}$. Show that P and P' both lie on a circle centre O, radius $\frac{AB\lambda}{1-\lambda^2}$. [Hint: Let M be the midpoint of P and P'; show that OM is perpendicular to BP.]

2 *Given a line segment of length 1, describe how to construct a line segment of length $\sqrt{2+\sqrt{3}}$ using just a straight-edge and a pair of compasses. [You do not need to prove that your construction works.]*

3 *(a) Show that the hyperbolic distance from the origin to the point x, for a positive real number x is $2\tanh^{-1} x$.*

(b) Deduce that the hyperbolic distance from z to w is $2\tanh^{-1}\left|\frac{z-w}{\overline{w}z-1}\right|$.

4 *Let ABC be a triangle with incentre I and inradius r. Let γ be a circle inside the triangle tangent to the sides AC and BC, and externally tangent to the incircle of $\triangle ABC$ (i.e. the incircle of $\triangle ABC$ and γ meet at a point T, where they have a common tangent, and the rest of γ lies outside the incircle). Let γ have radius r' and centre J. Show that $\frac{r'}{r} = \frac{1-\sin\frac{C}{2}}{1+\sin\frac{C}{2}}$, where C is the angle $\angle ACB$.*

5 *Find the area of the hyperbolic triangle with vertices at 0, $\sqrt{\frac{\sqrt{3}-1}{\sqrt{3}+1}}$ and $\sqrt{\frac{\sqrt{3}-1}{\sqrt{3}+1}}i$. [Hint: $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.]*

6 *Let Γ be a circle. Let A and B be points on Γ. Let C be a point on Γ, and D a point outside Γ such that $ABCD$ is a parallelogram. Extend the line DA to meet Γ again at X. Show that $BX = AC$ and $AB = CX$.*

7 *There is a semiregular polyhedron with 2 square faces, one triangular face, and one pentagonal face meeting at each vertex. How many:*

(i) triangular faces

(ii) square faces

(iii) pentagonal faces

(iv) vertices

(v) edges

does it have?

8 *Show that the area of an hyperbolic triangle with angles* α, β, *and* γ *is* $\pi - \alpha - \beta - \gamma$. *[You may use the fact that the area of a doubly asymptotic triangle with angle* θ *is* $\pi - \theta$.*]*

9 *(a) Show that inversion in a circle sends lines not passing through the centre of the circle to circles passing through the centre of the circle.*

(b) What are hyperbolic straight lines in the disc model? Prove your answer. [You may use the hyperbolic isometries taught in class without proof.]

10 *How many hyperfaces, faces, edges and vertices does a 4-dimensional hypercube have? Justify your answer.*

11 *Let* ABC *be the triply asymptotic hyperbolic triangle with vertices at* 1, $0.6+0.8i$ *and* w *where* w *is the point on the boundary of the unit disc such that* $0.5i$ *lies on the hyperbolic line between 1 and* w *. Find an hyperbolic isometry sending* ABC *to the hyperbolic triangle with vertices at the* 1, -1 *and* i.

12 *Find the endpoints of the hyperbolic line from* $0.5-0.5i$ *to* $\frac{1-5i}{13}$ *in the disc model. (i.e. find the points where this hyperbolic line meets the boundary of the disc.)*

13 *Describe the construction to trisect an acute angle using a straight-edge with a fixed distance marked on it and a pair of compasses, and prove that it does indeed produce an angle one third the size of the original.*