MATH 2051, Problems in Geometry
Fall 2007

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Mock Midterm Examination

Time allowed:

Section A – 45 minutes

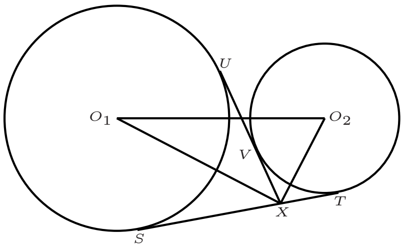
Section B – 45 minutes

Calculators not permitted.

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Section A

1 Let Γ_1 and Γ_2 be two disjoint circles with centres O_1 and O_2. Let l_1 be tangent to Γ_1 at S and tangent to Γ_2 at T, both on the same side of the line O_1O_2. Let l_2 be tangent to Γ_1 and Γ_2 at U and V respectively, on opposite sides of O_1O_2. Let l_1 and l_2 meet at X. Prove that $\angle O_1XO_2 = 90°$.



2 Let Γ_1 and Γ_2 be two circles that intersect at A and B. Let the tangent to Γ_1 at A meet Γ_2 again at P, and let the tangent to Γ_2 at B meet Γ_1 again at Q. Prove that AQ and PB are parallel.

3 Let ABC be a triangle, and let D, E and F be the feet of the altitudes from A, B and C respectively.

(a) Find the lengths DE, DF and EF, and the angles $\angle DEF$, $\angle DFE$ and $\angle EDF$.

(b) Prove that the ratio of areas $\frac{\triangle DEF}{\triangle ABC} = 2\cos A\cos B\cos C$. [You may use the facts that $\sin 2\theta = 2\sin\theta\cos\theta$, and $\sin\theta = \sin(180° - \theta)$.]

Section B

1 Let $ABCD$ be a square of side 1, and let P, Q, R and S be points on AB, BC, CD, and DA respectively, such that $PQRS$ is a rectangle (all angles $90°$), and $AS < AP$ (so $\angle APS < 45°$). Prove that $PQRS$ is a square.

2 Let $ABCD$ be a convex quadrilateral (all angles $< 180°$), such that the diagonals AC and BD meet at right angles at X. Let R and S be the feet of the perpendiculars from B to CD and DA respectively. Show that if $XS = XR$, then $AB.DC = BC.AD$.