MATH 2112/CSCI 2112, Discrete Structures I
Winter 2007
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Homework Sheet 3
Hints & Model solutions

Sheet 2

4(a)

$$\begin{array}{l} \phi(x) \\ \therefore (\exists y)(\phi(y)) \end{array}$$

$\phi(x)$	
$\phi(x) \vee \neg(\forall y)(\neg\phi(y))$	by generalisation
$(\forall y)(\neg\phi(y)) \to \phi(x)$	by $p \to q \equiv q \vee \neg p$
$(\forall y)(\neg\phi(y)) \to (\forall y)(\neg\phi(y))$	tautology
$(\forall y)(\neg\phi(y)) \to \neg\phi(x)$	by instantiation
$(\forall y)(\neg\phi(y)) \to (\phi(x) \wedge \neg\phi(x))$	by conjunction
$(\forall y)(\neg\phi(y)) \to 0$	by $p \wedge \neg p \equiv 0$
$\neg(\forall y)(\neg\phi(y))$	by $p \to 0 \equiv \neg p$
$\equiv (\exists y)(\phi(y))$	

5 *Another quantifier that is sometimes used is $(\exists! x)(\phi(x))$, meaning that there is exactly one x such that $\phi(x)$ is true. Rewrite this expression using $\exists$ and $\forall$.*

The easiest way is probably:

$$(\exists x)(\phi(x) \wedge (\forall y)(\phi(y) \to (y = x)))$$

Sheet 3

1 *Prove or disprove the following directly from the definitions:*

(a) For any odd positive integer n, at least one of the numbers n, $n+2$, $n+4$, and $n+6$ is composite.

This is false when $n = 1$ – none of 1, 3, 5, and 7 is composite, since if $ab = n$ where a and b are positive integers with $a \leqslant b$ (if $a \geqslant b$, and $ab = n$, then $ba = n$ and $b \leqslant a$, so any composite positive integer can be expressed as ab with $1 < a \leqslant b$) then $a^2 \leqslant n$. Therefore, when $n = 1$ or

$n = 3$, $a = 1$ is the only possibility, so 1 and 3 are not composite. When $n = 5$ or $n = 7$, $a = 1$ and $a = 2$ are the only possibilities. 5 and 7 are not even, since if $2b = 5$ then $2 < b < 3$, so b is not an integer. If $2b = 7$, then $3 < b < 4$, so b is not an integer. Therefore, $a = 1$ is the only possibility. Thus, none of 1, 3, 5, and 7 is composite, so the claim is false when $n = 1$.

(b) 113 is prime.

As above, if 113 is composite, it can be written as ab for positive integers a and b with $1 < a \leqslant b$, and therefore, $a^2 \leqslant ab = 113$. Therefore, we must have $a \leqslant 10$ as $11^2 = 121 > 113$. For $a = 2$ we would have $56 < b < 57$, so b is not an integer. For $a = 3$, $37 < b < 38$; for $a = 4$, $28 < b < 29$; for $a = 5$, $22 < b < 23$; for $a = 6$, $18 < b < 19$; for $a = 7$, $16 < b < 17$; for $a = 8$, $14 < b < 15$; for $a = 9$, $12 < b < 13$; and for $a = 10$, $11 < b < 12$. Therefore, the only value of a for which b can be an integer is $a = 1$. Therefore, 113 is not composite, so since $113 > 1$, 113 is prime.

(c) 142 is even

$142 = 2 \times 71$, so it is even by definition of even.

(d) For any integer n, if n is even, then so is $n^3 + 7n + 12$.

If n is even, then $n = 2m$ for some integer m. Therefore, $n^3 + 7n + 12 = (2m)^3 + 7(2m) + 12 = 8m^3 + 14m + 12 = 2(4m^3 + 7m + 6)$, and $4m^3 + 7m + 6$ is an integer, so $n^3 + 7n + 12$ is also even.

(e) There is a natural number n such that $n^2 + 7n + 12$ is prime.

This is false. $n^2 + 7n + 12 = (n + 3)(n + 4)$. As n is a natural number, $n + 3$ and $n + 4$ are positive integers that are greater than 1. Therefore, $(n + 3)(n + 4)$ can be written as a product of two integers greater than 1. Thus, it is not prime.

2 Prove or disprove the following. You may use the results proved in the course, or in earlier questions. You do not need to write out proofs that particular numbers are prime.

(a) If x is a rational number, and there is an integer n such that nx is an integer, then x must be an integer.

This is false. For example, let $x = \frac{1}{2}$, $n = 4$. Now $nx = 2$, which is an integer. However, x is not an integer.

(b) There are natural numbers m and n such that m, n, and $m+n$ are all prime.

This is true, for example $m = 2$, $n = 3$. Then $m + n = 5$ is prime.

I intended to set the question with the additional condition that m and n should both be at least 100 (though both at least 3 would work just as well) in which case the statement would be false, because of the following argument:

Let m and n be prime numbers that are both at least 100. If $m = 2k$, then k is at least 50, and therefore, is greater than 1, so m can be written as a product of positive integers greater than 1. This is impossible, as m is prime. Therefore, m cannot be even. Thus, m must be odd. For the same reason, n must be odd. Therefore, their sum must be even, so $m + n = 2l$ for some integer l, which must be at least 100, because $m + n \geqslant 100$. Therefore, $m + n$ is not prime, so the claim is false.

(c) If x and y are rational numbers, and n is an integer such that nx is an integer, and ny is an integer, then $n^2(x^2 + y)$ is an integer.

$n^2(x^2 + y) = n^2x^2 + n^2y = (nx)^2 + n(ny)$. Since the square of an integer is an integer, $(nx)^2$ is an integer. Also, $n(ny)$ is the product of the two integers n and ny, so it is an integer. Finally, $n^2(x^2 + y)$ is the sum of the integers n^2x^2 and n^2y, so it is an integer.

(d) All numbers of the form $12k + 5$, where k is a natural number between 0 and 4, are prime.

The numbers of this form are: 5, 17, 29, 41 and 53, and these are all prime, so the statement is true.

(e) If $n > 2$ is a positive integer, then at most 3 of $10n$, $10n + 1$, ..., $10n + 9$ are prime.

This is false. If $n = 10$, then 101, 103, 107 and 109 are all prime.

(f) There is a positive integer n such that n, $n + 1$, $n + 2$, ..., $n + 100$ are all composite.

Hint:

Try to construct this number n so that n is divisible by 2, $n+1$ is divisible by 3, $n+2$ is divisible by 4, and so on up to $n+100$ is divisible by 102.